Earnings and Firm Value in the Presence of Real Options

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ABSTRACT: This paper examines the relationship between earnings and firm value in a dynamic real-options setting in which firms respond to a continuous stream of investment and disinvestment opportunities. Built only on the basic assumptions that firms can initiate and abandon investments at their discretion and that firms seek to maximize value creation, the model yields a non-monotonic and highly nuanced mapping from earnings to firm value that differs markedly from the convex, monotonic relationship obtained in static option models. The model predicts the empirically documented phenomenon that firm value can decrease in earnings among loss-making firms and further implies that the value of future growth opportunities decreases when earnings are high. At a more general level, dynamic real options make the computation of firm value from earnings coarse in the sense that both firm value and current earnings are simultaneously determined by the past interaction of opportunities and decisions, so that valuation amounts to an inference based on an endogenously created correlation between earnings and value, rather than on an immediate causal link.

1. Introduction

Accounting-based valuation models have emphasized the role of earnings in predicting the economic value generated by firms' activities. Accordingly, the relationship between earnings and market values or stock returns has received much attention in past research. The foundational work by Ohlson (1995) and Feltham and Ohlson (1995, 1996) employs a stochastic linear information model in which firm value takes the form of a linear function of the firm's current earnings (and possibly other information). A known limitation of this result is that it arises in a world in which firms do not take corrective action as new information arrives over time, so that the symmetry of the information structure translates into linearity of the valuation function.[1] A number of models have been proposed over the years that incorporate opportunities for corrective action, also known as real options, into the valuation problem.[2] Consistent with option pricing theory, these models generally imply that the mapping from earnings to firm value should be convex rather than linear, a result that essentially reflects the behavior of standard option pricing formulas, with earnings treated as the underlying (Burgstahler and Dichev 1997; Zhang 2000; Ashton, Cooke and Tippett 2003; Chen and Zhang 2007).

The purpose of this paper is to show that the relationship between earnings and firm value is neither linear nor convex, and generally not even monotonic, if one considers real options not as a static endowment of option securities but as a continuous flow of opportunities to which firms respond in real time. This is true even if earnings have no peculiar informational properties vis-à-vis plain cash flows, and the model will indeed not assume any such properties. At a conceptual level, the critical point is that treating options as a static endowment, with earnings as the underlying, fails to recognize that earnings are, first and foremost, themselves determined by past option exercises. In a static model, option exercises only occur in the future and therefore do not affect current earnings. In a dynamic setting, by contrast, the business opportunities that real options reflect can arrive, change in value, and be taken advantage of at any time. In this case, the exogenous arrival of opportunities and the endogenous option exercise strategy firms adopt in response simultaneously determine both the firm's current earnings and the value of its extant

[1] For example, the linear valuation formula implies that firms never declare bankruptcy even if earnings should evolve unfavorably and turn into large losses. In practice, by contrast, most businesses fail within a few years.
[2] Real options arise when outcomes are uncertain, decision are, at least in part, irreversible, and firms have discretion, particularly with respect to timing, over their actions (Dixit and Pindyck 1994).

options. As a result of this joint determination, earnings and firm value show predictable correlation, but this correlation is not indicative of a direct causal link between the two.

As a simple, heuristic illustration, consider the opportunity to open a new store. As long as the company holding the option does not act, no earnings are generated but the option has some value. Once the company decides to open the store, earnings are generated but the option value disappears because the option has been exercised. Not only are earnings nothing like an underlying of the option in this example, but the relationship between earnings and option value is decreasing, contrary to what a standard call option formula, evaluated with earnings as an underlying, would imply. To obtain a flavor for the potential additional intricacies of this valuation problem, one might also consider how sales at the company's existing stores could suffer if the new store is opened, how sales trends at these existing stores are correlated with the value of the option and thus the likelihood of its exercise, and how opening the new store would affect the company's opportunities to open further stores in other locations.

The model in this paper aims for a compromise between permitting these considerations to affect firm value and maintaining analytical tractability. The two critical properties of the model are that earnings are the result of firms' taking action in response to opportunities, and that actions taken today affect the availability of opportunities tomorrow. Earnings are thus no longer the mechanical product of an exogenous information flow. Rather, the information flow in the model, while consistent with the linear information structure of the Feltham-Ohlson framework, reflects the arrival of opportunities to which the firm can respond at its discretion, and earnings are the joint outcome of these exogenous opportunities and firms' endogenous response strategy. In particular, the firm is given options to undertake new investments and to abandon prior investments, in analogy to exercising call and put options, and exercising any of these options today both affects future earnings and reduces the number of options left for the future.

The results of the analysis point to the limitations of earnings as a valuation input but also provide some potentially useful insights how existing valuation frameworks may be refined. In a dynamic setting with a continuous flow of opportunities, inferences about firm value based on earnings are inherently coarse in the sense that any given earnings number can rationalize a range of past decision histories. The reason is that earnings are not the primitive source of value but rather provide information about past actions and opportunities, which in turn determine future opportunities and value flows. For example, an observed increase in earnings can arise because exogenous events have increased the payoff to the firm's existing investments. Alternatively, the earnings increase might mean that the firm has exercised one of its growth options and created a new investment that now improves its profits. Yet a third explanation may be that the firm has eliminated a previously loss-making investment. The implications with respect to firm value differ considerably across these scenarios.

Despite this coarseness of the inference, several notable regularities arise. First, both the value of investments in place and the value of future growth opportunities can decrease in earnings among loss-making firms. While counterintuitive on the surface, this result is consistent with empirical observations by, among others, Hayn (1995) and Collins, Pincus and Xie (1999), and follows from the logic that firms continue loss-making operations only if the intrinsic value of the underlying investment is still positive. Firms with larger losses, on average, are carrying more such investments than firms with earnings around zero. Further, the growth option value is not monotonically increasing in earnings even among profitable firms but can have multiple local minima and maxima. In particular, the value of future growth opportunities declines when firms have very high earnings, as high earnings imply that firms may have harvested a large share of their investing opportunities already, leaving fewer investing options for the future.

In the context of business valuation, the continuous arrival of opportunities, the feedback effect of endogenously determined exercise strategies, and the possibility that opportunities and actions can occur in the past present difficulties absent in the pricing of standalone option securities. A number of existing option-based valuation models have therefore simplified the problem to a static setting in which the firm's current opportunities do not depend on its past actions. The models by Burgstahler and Dichev (1997), Ashton, Cooke and Tippett (2003) and Ataullah, Rhys and Tippett (2009) permit firms to switch from receiving earnings from existing operations to receiving an alternative form of payment.[3] The switching opportunity corresponds to a put option with earnings as an underlying, from which the convexity property follows naturally. Zhang (2000) adds a call option feature that permits firms to expand the asset base that generates the earnings. The main conclusion from static option models is that, consistent with the behavior of option pricing formulas, firm value becomes an increasing, convex function of current earnings.

[3] The alternative payment is typically interpreted as resulting from redirecting the firm's assets to a different use by taking discretionary actions such as divestitures, restructurings, or mergers.

There are two broad classes of models that do give consideration to the intertemporal dependence of actions and opportunities. Berk, Green and Naik (1999) consider the firm as a series of investment projects that arrive randomly over time. At a conceptual level, though not in model setup and research objective, the model presented hereafter belongs to this category.[4] An alternative class of models is built around the notion of dynamic capacity adjustment, based on a parametric structure specifying production technologies and the evolution of input and output prices. An early example is Abel, Dixit, Eberly and Pindyck (1996), which Gietzmann and Ostaszewski (2004) adapt to the earnings-based valuation problem to develop an alternative to the residual income approach.[5] More recently, Livdan and Nezlobin (2016) employ a capacity model with growth options to examine the effect of various methods of calculating book value on the precision of equity valuation.

The aim of this paper is not to compare different methods of value measurement. Rather, the objective is to determine what relationship one should expect between current value flows, whether measured by cash flows, accrual-basis earnings, residual income or otherwise, and the intrinsic value of the business entity as a whole in the presence of real options, absent any measure-specific artifacts. Hence, the model makes no distinction between cash flow and accrual-basis earnings. This is not done in ignorance of the difference between the two but, au contraire, serves to highlight that phenomena that one might be tempted to ascribe to the peculiarities of accrual accounting rules can obtain even in their absence. The model is an exercise in applying the lex parsimoniae in that all results ultimately rest on two fundamental assumptions: opportunities exist to initiate and abandon investments discretionally, and firms seek to maximize the value generated by their investments. This approach is similar in spirit to Hemmer and Labro (2016), who obtain an S-shaped market response to earnings in a setting without bias or noise in accounting.[6] Invariably, this cannot rule out other possible explanations for the joint behavior of earnings and firm value, such as uncertainty about the precision of earnings information (Penno 1996; Subramanyam 1996), but it provides a parsimonious alternative.

[4] The focus of the analysis by Berk, Green and Naik lies on the properties of risk and return rather than on the relationship between earnings and value.
[5] The objective of the original Abel et al. paper is to explain the role of uncertainty in investing incentives, not to provide a valuation model.
[6] Hemmer and Labro model a firm whose manager optimizes myopically over a binary switching option each period.

2. Model Setup

The model describes a firm that generates value by undertaking investments. An investment is defined as a stream of net cash flows initiated at a particular point in time. Investments should be thought of as the smallest unit by which the firm can scale its operational activities. What corresponds to an investment in practice depends on the firm's business model and can range from hiring individual employees or adding individual pieces of production equipment to introducing new products, setting up store locations and production sites, or expanding operations to new geographic areas or market segments.

The values of existing and potential new investments change stochastically as time passes. The firm can make new investments and terminate existing ones in response to these value changes, akin to holding and exercising call and put options. The firm's earnings are defined as the aggregate net cash flow from all active investments at a given time. This simplification of the accounting to a cash-basis is made on purpose to stress that none of the results to be derived depend on the idiosyncrasies of accrual-basis accounting. Agency conflicts or other frictions are not the subject of study, and hence the firm is assumed to follow investment and termination policies that maximize total intrinsic enterprise value.

The following notation will be used throughout the text. Let $n(t) \in \mathbb{N}$ denote the number of investments that the firm, as of time t, has made since its founding date, which is normalized to $T_0 \equiv 0$. The initiation date of investment $i = 1,2,\dots,n$ is denoted by T_i. From time T_i onward, each investment i generates a continuous net cash flow Y_i. As long as the firm does not liquidate the investment, the cash flow follows the stochastic process

$$dY_i(t) = \sigma\, dZ_i(t) \tag{1}$$

where $\sigma \in \mathbb{R}^+$ and Z_i is standard Brownian motion.[7] The Z_i of different investments are uncorrelated.[8] If the firm liquidates the investment at some date $\bar{T}_i$, the cash flow becomes $Y_i(t) = 0$ at

[7] The economic meaning of σ is twofold. A high value may correspond to large-scale operations typical of, for example, oil companies or steelmakers, while a low σ might characterize firms that can scale operations at a smaller level, such as personal service and retail businesses. The second meaning of σ is business risk in that higher σ implies greater cash flow volatility. For example, microchip manufacturers likely face higher σ than electric utilities.
[8] Nonzero correlation between the Z_i would improve the information content of the firm's total earnings but would not alter the qualitative insights from the model. Similarly, making σ the same for all investments is done for ease of exposition but is not essential for any results.

all $t > \bar{T}_i$, i.e., the liquidation decision is irreversible.[9] Net liquidation proceeds are normalized to zero.[10] The firm's earnings P, i.e., the total net cash flow of all active investments, are thus

$$P(t) = \sum_{i=1}^{n(t)} Y_i(t) \tag{2}$$

The firm can make new investments at any time. The initial cash flow of an investment i added at time T_i is $Y_i(T_i) = X(T_i)$, where X is a state variable that reflects present investment opportunities. When no new investment is made, the evolution of X follows the process

$$dX(t) = \omega\, dZ(t) \tag{3}$$

where $\omega \in \mathbb{R}^+$ and Z is standard Brownian motion that is uncorrelated with the Z_i of any existing investments.[11] (The topic of correlation is revisited during the discussion in Section 3.) Each time the firm initiates a new investment, X additionally experiences an instantaneous downward jump by $\mu \in \mathbb{R}^+$. The value of the opportunity process at time t can therefore be written as

$$X(t) = X\big(T_{n(t)}\big) - \mu + \int_{T_{n(t)}}^{t} \omega\, dZ(s) \tag{4}$$

The economic rationale behind (4) is central to the theme of the paper, namely that present decisions affect future opportunities. To understand the motivation, consider the simpler structure obtained by setting $\mu = 0$, in which case X is exogenously determined by (3) only. The result would be patently unrealistic. The most obvious reason is that, in reality, any given opportunity can only be taken advantage of once, so that an investment made today, ceteris paribus, mechanically reduces the number of opportunities left for tomorrow. Leaving X unaffected by the decision to invest would then be equivalent to permitting the firm to implement the same investment an arbitrary number of times. A further reason for $\mu > 0$ is that past and future investment opportunities can modify each other. For example, making a new investment may not only generate additional profit by serving customers who previously did not buy the firm's products but may also have a detrimental effect on the profits from previous investments (say, because of

[9] The option to reinstate the investment at a later date would only create additional complexity without altering any conclusions. For an analysis of the reinstatement option, see Dixit (1989).

[10] The effect of strictly positive or negative liquidation values are straightforward but have no bearing on the conclusions of the paper. In practice, the net liquidation value is the amount of proceeds from selling any remaining assets pertaining to the investment, less the amount of liquidation costs, such as severance pay and legal fees, and this net amount can be positive or negative.

[11] Both (1) and (3) are consistent with the Feltham-Ohlson framework, or, more closely, with its continuous-time variant developed by Govindaraj (1992).

product cannibalization) and on the availability of future investment opportunities (because introducing a new product today means there will be fewer potential customers left to buy a second new product introduced tomorrow). Moreover, undertaking investments might also increase the firm's experience and efficiency and strengthen customer relationships, which could impact future opportunities favorably. The structure in (4) aims for a balance between tractability and incorporating these intertemporal tradeoffs.

The magnitude of μ determines the extent to which taking advantage of an investment opportunity today reduces the set of investment opportunities left for tomorrow. As an illustration, consider an oil company contemplating the construction of a new refinery in a region currently undersupplied with petroleum products. Since demand for petroleum products tends to grow gradually over time (say, as motorization increases or the total population grows), making the investment today likely means that the opportunity to build a second refinery will not be forthcoming anytime soon, i.e., μ is likely to be high in this case. Contrast this scenario with the case of a developer of mobile devices who is considering the introduction of a new generation of its product. Since technology changes fast in this industry and products therefore have a relatively short life cycle, investing in the development of a new product generation today is unlikely to have a lasting impact on opportunities to invest in the development of new products in the future, i.e., μ is more likely to be low in this case.[12]

3. Earnings and Firm Value in the Presence of Real Options

The primary object of interest is the contemporaneous relationship between earnings and firm value. The analysis progresses through three incrementally complex variants of the model. A simplified, partially deterministic variant in Section 3.1 considers a fixed history of investments with abandonment options. A fully stochastic scenario in Section 3.2 additionally features the option to undertake new investments. Section 3.3 extends the model to a setting in which generating opportunities to undertake new investments is costly.

3.1. Earnings and Firm Value in the Presence of Abandonment Options

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[12] A related interpretation of μ is product substitutability or cannibalization. For example, a pharmaceutical company marketing a brand of cough medication in capsule form will face a higher μ when introducing a new cough syrup than it would if it introduced an antidepressant instead.

In this simplified variant of the model, the initiation of new investments is treated as deterministic. In particular, suppose that the firm makes n investments by time t. These investments have exogenously fixed initiation dates $\boldsymbol{T} = (T_1, \dots, T_n)$, with $T_n < t$, and their payoff processes are given by (1). For now, why and how the firm decides to make these investments shall be ignored, and their initial cash flows $Y_i(T_i) = \bar{Y}$ are treated as a deus ex machina. The firm can abandon any of these n investments at any time after their respective initiation dates.

The optimal abandonment strategy can be intuited from the following heuristic. The structure of the cash flow in (1) implies that, conditional on any given future abandonment strategy, the expected future payoff from investment i is increasing in the current cash flow level $Y_i(t)$. Hence, if abandonment is optimal at some $Y_i = \underline{Y}$, it must also be optimal when $Y_i < \underline{Y}$, and, indeed, the value-maximizing decision rule is to abandon the investment at the first instance when its cash flow declines to a threshold level $\underline{Y}$.[13] Formally, this decision rule and the corresponding intrinsic investment value, hereafter denoted by v, follow from a standard option pricing program, whose solution is given below. All proofs can be found in the Appendix.

Lemma 1. *The firm optimally abandons investment i at time $\bar{T}_i = \inf\{s: Y_i(s) \leq \underline{Y}\}$, where*

$$\underline{Y} = -\frac{\sigma}{\sqrt{2r}}$$

The investment value associated with this abandonment strategy is

$$v(Y_i) = \frac{Y_i}{r} + \frac{\sigma}{\sqrt{2r^3}} e^{\frac{\sqrt{2r}}{\sigma}(\underline{Y} - Y_i)}$$

when $t \leq \bar{T}_i$ and zero otherwise.

The properties of v and $\underline{Y}$ in Lemma 1 are well-known in option pricing theory, but two observations deserve explicit mentioning here because they lay the foundation for all subsequent results. First, the investment value v is monotonically increasing and convex in the underlying cash flow Y_i as long as the investment remains active. At the abandonment date $\bar{T}_i$ (which is a random variable), the investment value drops to $v(\underline{Y}) = 0$ and remains there indefinitely. Second, the abandonment threshold of $\underline{Y} < 0$ means that, at any $Y_i \in (\underline{Y}, 0)$, the firm is willing keep

[13] Consistent with the empirical evidence by Berger, Ofek and Swary (1996), a positive (negative) liquidation value would increase (decrease) $\underline{Y}$ and v but not alter the directionality of the following results.

the investment alive even in the face of a negative current cash flow. The reason is that the option to abandon the investment at a later date still holds enough value to outweigh the current losses.[14] Taken together, these two observations imply that an abandoned investment contributes a higher cash flow to the firm's earnings, namely zero, than an active but loss-making investment. Yet, the loss-making investment has a strictly positive intrinsic value while the abandoned investment is worth zero.

This conclusion has important consequences for the relationship between earnings and firm value. The scenario $n = 1$, in which the firm only makes a single investment at some date T_1, delivers an intuitive illustration. The firm's earnings at time $t > T_1$ then simplify to $P(t) = Y_1(t)$, and total firm value, hereafter denoted by m, becomes

$$m(Y_1; t) = v(Y_1)\mathbf{I}_{t\le\bar{T}_1} = v(P)\mathbf{I}_{t\le\bar{T}_1} \quad (5)$$

where the indicator function $\mathbf{I}_{t\le\bar{T}_i}$ takes the value zero if investment i has been abandoned by time t. In view of Lemma 1, m is increasing and convex at all $Y_1 \in (\underline{Y}, 0) \cup (0, \infty)$ but has a singularity at $Y_1 = 0$. Although a cash flow of $Y_1 = 0$ can be generated by either an active or an abandoned investment, the prior probability that an active investment produces a cash flow of exactly zero is zero while there is a strictly positive probability of abandonment at all $t > T_1$. Thus, for any firm with $n = 1$ and earnings of zero,

$$\Pr(t > \bar{T}_1 | P(t) = 0) = 1$$

which implies $\mathbf{I}_{t\le\bar{T}_1} = 0$ and thus $m = 0$. In visual terms, if one were to plot the earnings and firm values of a large sample of ex-ante identical, single-investment firms at time t, one would obtain a mass point at $(P, m) = (0,0)$ but strictly positive m at both $P > 0$ and $\underline{Y} < P < 0$. In other words, with only one investment in play, firm value is non-monotonic in earnings.

The realism and practical usefulness of the single-investment scenario leave much to be desired, as a firm that shuts down its only investment project likely ceases to exist as a going concern, so that the mass point at $(P, M) = (0,0)$ may not appear in any earnings dataset, not to mention that most firms make more than one investment to begin with. More importantly, the non-monotonicity in the $n = 1$ case takes the form of a point discontinuity at $P = 0$. For $n > 1$,

[14] The negative abandonment threshold is a classic result in real options theory (Dixit and Pindyck 1994) and obtains even if the investment has a moderately positive liquidation value or if the cash flow process is subject to a negative drift over time. The firm's loss tolerance and the investment's intrinsic value increase in the volatility parameter σ and decrease in the discount rate r, consistent with the idea that the value of an option increases in the volatility of the underlying and decreases if the potential future benefit of continuing is diminished by higher discounting.

the probability of abandoning all investments by time t and thereby attaining the discontinuity point decreases. It therefore remains to be established whether the one-investment scenario has broader relevance.

In case of multiple investments, both earnings and firm value are linear aggregations of their investment-level counterparts, with earnings given by P in (2) and firm value given by

$$m(\boldsymbol{Y};t) = \sum_{i=1}^{n} v(Y_i)\mathbf{I}_{t\leq \bar{T}_i} \tag{6}$$

where $\boldsymbol{Y} = (Y_1, \dots, Y_n)$. Unlike in the case $n = 1$, the aggregate earnings value P from $n > 1$ investments is not a sufficient statistic for the set of investment cash flows $\boldsymbol{Y}$ with respect to calculating m. Hence, firm value calculated from earnings takes the form of a statistical inference

$$\begin{aligned}\widehat{m}(P;t) = E(m(\boldsymbol{Y};t)|P) &= \sum_{i=1}^{n} E\left(v(Y_i)\mathbf{I}_{t\leq \bar{T}_i}\middle|P,t\right)\\ &= \frac{P}{r} + \frac{\sigma}{\sqrt{2r^3}}\sum_{i=1}^{n} E\left(e^{\frac{\sqrt{2r}}{\sigma}(\underline{Y}-Y_i)}\mathbf{I}_{t\leq \bar{T}_i}\middle|P,t\right)\end{aligned} \tag{7}$$

where the right-hand side follows from Lemma 1. There is no tractable expression for (7) in terms of elementary functions but, as the following result demonstrates, the logic behind the single-investment scenario can be shown to carry through.

Proposition 1. *For sufficiently large σ or t, firm value is decreasing in earnings around $P = i\underline{Y}$, for any $i = 1, \dots, n-1$ and $n > 1$ and any investment history $\boldsymbol{T}$.*

The gist of Proposition 1 is that, among firms incurring losses, firm value can be a decreasing function of earnings on as many disjoint intervals as the original number of investments the firm has made. The intuition behind the mechanics of this result is, in principle, a generalization of the observations made about the special case $n = 1$. Consider two earnings values P and $P + \varepsilon$, for some $\varepsilon > 0$. By a somewhat stylized heuristic, there are two possible explanations for the difference ε. The first is that, ceteris paribus, some of the investments underlying $P + \varepsilon$ produce higher cash flows than the investments underlying P. In this case, firm value m is higher given $P + \varepsilon$. The second explanation is that, ceteris paribus, P contains an additional investment that currently generates a cash flow of $-\varepsilon$. By revealed preference, this additional investment must have positive intrinsic value, or else the firm would have abandoned it. Hence, firm value is higher given earnings of P in this case.

Proposition 1 is essentially a statement about the relative likelihood of these two explanations as a function of P and ε. The core of the argument is that large losses can only come from a large number of investments, whereas earnings of smaller magnitude are more likely to be the output of a small number investments. If, for example, $P = i\underline{Y} - \frac{\varepsilon}{2}$, it must be true that P has at least $i + 1$ underlying active investments because otherwise there would exist at least one investment with a cash flow below $\underline{Y}$, which would be inconsistent with the firm's optimal abandonment strategy. In view of Jensen's inequality, firm value is therefore at least

$$\widehat{m}(P;t) \geq nv\left(\frac{n-1}{n}\underline{Y} - \frac{\varepsilon}{2n}\right) \tag{8}$$

At earnings of $P + \varepsilon = i\underline{Y} + \frac{\varepsilon}{2}$, on the other hand, it is possible that only i investments have survived.[15] The probability that the cash flow of any investment declines to the abandonment threshold $\underline{Y}$ by time t increases in t and in the volatility parameter σ. Hence, that $P + \varepsilon$ has indeed no more than i underlying active investments becomes increasingly likely as t and σ increase, so that one can eventually bound firm value by

$$\widehat{m}(P+\varepsilon;t) \leq v\left(\underline{Y} + \frac{\varepsilon}{2}\right) + \varepsilon \tag{9}$$

For small enough ε, (8) exceeds (9) for any number of $i = 0,1,\dots,n-1$.

To obtain a clean analytical conclusion, the argument sketched out above relies on setting t or σ to sufficiently large values, equivalent to requiring a certain minimum level of cash flow volatility. To gain some intuition whether low cash flow volatility would reverse the result, consider the extreme case of $\sigma = 0$. The abandonment threshold then becomes $\underline{Y} = 0$ and the cash flow of each investment degenerates to a deterministic perpetuity $Y(t) = \bar{Y}$ at all t. No firm ever incurs losses in this world, and hence the claim that firm value increases in losses remains vacuously true. While lack of tractability limits the formal claims that can be made for small values of σ, simulations suggest that the claim of Proposition 1 applies even if σ and t are strictly positive

[15] The argument illustrates why Proposition 1 requires negative earnings: the observation that

$$\Pr\left(\sum_{i=1}^{n} \mathbf{I}_{t\leq\bar{T}_1} > \frac{P(t)}{\underline{Y}}\right) = 1$$

only leads to a narrowing-down of the possible number of investments if $P < 0$. For $P > 0$, $\widehat{m}$ is increasing in P.

but of small magnitude, i.e., large values of t or σ are sufficient but not necessary conditions for Proposition 1. Figure 1 provides a graphical illustration based on a simulation of the model.

While the formal argument may appear somewhat abstract, the economics of the possibility that firm value declines as earnings increase are quite straightforward. A retail company operating department stores in various locations might serve as an example. Suppose that each store location corresponds to a separate investment. Hoping that sales might recover, the company would keep a store open and operating for some time even if the store's profits are negative but, in line with Lemma 1, would eventually shut it down if the losses become unsustainably large. Then a retailer operating ten stores, all with an identical (and yet sustainable) level of losses, would report negative earnings equal to ten times the store-level cash flow and have a total firm value equal to ten times the store-level intrinsic value. If the cash flow in five of the ten stores now dropped to the abandonment threshold and these five stores were closed (with the other stores' losses remaining unchanged), both the amount of negative earnings and the amount of firm value would be cut in half. That the stores incur losses is essential to the argument: if the original ten locations had produced positive cash flows and five of them were later shut down due a deterioration in performance, earnings and firm value would decline together.

Empirical research has repeatedly documented an about-zero or negative correlation between earnings and firm value among loss-making firms (Hayn 1995; Burgstahler and Dichev 1997; Collins, Pincus and Xie 1999). The non-monotonicity result in Proposition 1 provides a plausible rationalization of this phenomenon in terms of standard business economics, and Figure 1 indeed shows some noticeable resemblance to a corresponding data plot by Burgstahler and Dichev (1997, Figure 3), which is based on actual data. The only two assumptions required for Proposition 1 are that firms are rational, profit-maximizing decision-makers and have the option to liquidate their investments at their discretion.[16] As further analysis will demonstrate, adding growth opportunities to the model reinforces the result.

It is important to emphasize that, despite the focus on $n > 1$ in Proposition 1, the multiplicity of investments is not at all critical to the non-monotonicity result. Even the one-investment case $n = 1$ shows non-monotonicity, albeit in degenerate form, around the singularity $P = 0$. That the

[16] Subramanyam (1996) rationalizes a non-monotonic relationship not based on optionality but based on the idea that earnings of large magnitude convey less precise information about firm value than earnings closer to zero. In contrast to Proposition 1, firm value can have at most one maximum on negative earnings under this setup.

abandonment option models by Burgstahler and Dichev (1997) and Zhang (2000), both of which treat the case $n = 1$, yield a globally convex and monotonic mapping from earnings to firm value is therefore not a consequence of the number of options in play. Rather, the difference between their results and Proposition 1 has two sources. First, their models value the firm conditional on $t < \bar{T}_1$, i.e., the possibility of a past exercise of the abandonment option plays no role. Second, the two models consider abandonment options that, in analogy to European put options, can only be exercised at a single, pre-specified date in the future rather than at any time of the firm's choosing.[17] The optimal exercise threshold of these European options is $\underline{Y} = 0$, which coincides with the earnings generated by abandoned investments. Either model feature makes the singularity disappear, and one obtains a globally convex and monotonic earnings-value relationship. Adding more abandonment options to the setup does not change any part of this reasoning.[18]

The importance of option style and the possibility of past exercises for Proposition 1 deserves more careful elaboration. Past abandonment decisions and the value of future abandonment opportunities are not independent of each other, for the simple reason that an abandonment option exercised yesterday is no longer available tomorrow. Conditioning on $t < \bar{T}_1$ implies a static setting in the sense that the firm is endowed with a set of opportunities as of the current date but has no past. In a setting with 'European style' options, this amnesia of sorts does no harm to the qualitative behavior of the earnings-value relationship because investments are eliminated precisely when both their intrinsic value and their underlying cash flow are identically zero, so that global monotonicity and convexity obtain regardless. If, on the other hand, the firm is free to terminate investment projects whenever it prefers, as seems realistic, the optimal decision strategy is to keep investments alive until their losses have reached a certain magnitude, as would not happen with European options. Past abandonment decisions now increase current earnings while coinciding with the elimination of previously valuable investment projects.

An additional point worth emphasizing is that no part of Proposition 1 depends on how much information those valuing the firm actually possess. Regardless whether only P is known or the

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[17] Similarly, Hayn (1995) models firm value as a piece-wise linear function equal to some constant multiple of earnings when this multiple exceeds the liquidation value and equal to the liquidation value otherwise. This representation which is the limit case when the abandonment option only exists over a very short future horizon.

[18] Zhang (2000) constructs a setting in which negative earnings and firm value can grow in magnitude simultaneously if earnings are calculated by conservatively biased accounting rules. Proposition 1 does not rely on any form of distortion.

project cash flows Y_i are observable individually, the average firm value, as a function of earnings, behaves in line with Proposition 1 if one examines earnings and firm values in a large sample of ex-ante identical firms. Further, it is neither necessary that new investments are made deterministically, as the next section will demonstrate, nor that cash flows are uncorrelated across investments. The extreme case of perfect positive correlation would, in fact, reduce the problem to an equivalent of the special case $n = 1$.

Proposition 1 may offer a new role for the balance sheet in valuation. Non-monotonicity in the earnings-value relationship obtains because the average number of active investments, and thus the size of a firm's current operations, is a non-monotonic function of earnings. The size of the balance sheet, on the other hand, is increasing monotonically in the size of operations. For example, in the case of the retail company discussed earlier, the amount of inventory and fixed assets would increase in the number of open stores because the assets of abandoned stores are disposed of and removed from the balance sheet. Including the balance sheet in $\widehat{m}$ would therefore amount to conditioning the inference on the number of active investments, which yields

$$\widehat{m}(P; n^*, t) = \frac{P}{r} + \frac{\sigma n^*}{\sqrt{2r^3}} E\left(e^{\frac{\sqrt{2r}}{\sigma}(\underline{Y} - Y_i)} \middle| P, n^*, t \right)$$

where

$$n^*(t) = \sum_{i=1}^{n} \mathbf{I}_{t \leq \bar{T}_i}$$

is the number of surviving investments at time t. One can readily verify that $\widehat{m}$ is now monotonically increasing in P. Collins, Pincus and Xie (1999) indeed observe a negative correlation between earnings and book value among loss firms, and they demonstrate empirically that the negative correlation between earnings and firm value disappears in a linear valuation model if firms' book values are included as value predictors.[19] The option value component of $\widehat{m}$ declines as P increases, and hence the inclusion of book value should make a greater impact on valuation among firms with low earnings. A related empirical finding by Barth, Beaver and Landsman

[19] In explaining the result, the authors hold that the value of loss firms reflects either positive future profits, if a return to profitability is anticipated, or the firm's liquidation value, if no such return is in sight. They conjecture that negative earnings have little informational value regarding these two future prospects, while book value does. This explanation is at odds with Lemma 1: an investment's current payoff, even if negative, always maps into intrinsic value, which includes expected future payoffs as well as the option value associated with liquidation.

(1998) is that the magnitude of the coefficient on book value in estimating firm value increases toward a firm's bankruptcy date.

3.2. Stochastic Investment Opportunities and Growth Options

A limiting feature of assuming a fixed investment history is the static nature of the investment decisions. It remains to be determined whether the logic underlying the non-monotonicity of firm value in earnings survives if investments are made strategically in response to stochastic opportunities. To this end, the option to make new investments, as formalized in the opportunity process $X(t)$ in (3) and (4), is taken into consideration from hereon. Stochastic growth opportunities have two implications with respect to firm value. First, the number of investments made in the past, $n(t)$, and their initiation dates, $\boldsymbol{T} = (T_1, \dots, T_n)$, become random variables, which adds a layer of uncertainty to an earnings-based inference about the intrinsic value of the firm's investments in place. Second, firm value now consists not only of the intrinsic value of existing investments but also of the growth option value associated with anticipated future investment.

The value implications of growth opportunities depend on the firm's investing strategy, which can be optimized by the following rationale. Recall that a decision to invest at time t means initiating a new cash flow process i with starting value $Y_i(t) = X(t)$ while incurring an instantaneous drop in X by $\mu > 0$ that reflects how investing today affects opportunities in the future. The firm is therefore indifferent between investing and waiting at a given value of X if

$$g(X) = g(X - \mu) + v(X) \tag{10}$$

where v is the new investment's intrinsic value, as derived in Lemma 1, and g denotes the (yet to be determined) growth option value.[20] At an intuitive level, deciding when (10) holds is a straightforward cost-benefit tradeoff: the firm may delay investment in the hope of obtaining a higher value of X in the future but then incurs the opportunity cost of not receiving any cash flow in the meantime.[21] The value of the forgone cash flow on the cost side is monotonically increasing in X. Then if the marginal benefit of waiting any longer is decreasing in X at the same time, it stands to reason that the firm's optimal investing strategy takes the form of a threshold value $\bar{X}$

[20] A similar indifference condition obtains in the dynamic investment model by Livdan and Nezlobin (2016), who interpret investing as adding capacity to a concave production technology.

[21] That investing at present diminishes future opportunities via the decrement μ is essential for the validity of this logic. If $\mu = 0$, there would be no well-defined notion of optionality because the possibility of waiting for a better X would hold no value, and hence there would be no connection between current and future opportunities.

such that the firm delays investment for as long as $X < \bar{X}$. Lemma 2 below gives the formal solution to the problem.

Lemma 2. *If the investment value $v(X)$ is logarithmically concave, the firm optimally initiates investment i at time $T_i = \inf\{s: X(s) \geq \bar{X}, s > T_{i-1}\}$, where $\bar{X}$ is the unique solution to*

$$\frac{v'(\bar{X})}{v(\bar{X})} = \frac{\sqrt{2r}}{\omega}$$

The growth option value associated with this investing strategy is

$$g(X) = \frac{e^{\frac{\sqrt{2r}}{\omega}(X-\bar{X})}}{1 - e^{-\frac{\sqrt{2r}}{\omega}\mu}} v(\bar{X})$$

Straightforward algebra shows that the investment value v in Lemma 1 meets the logarithmic concavity criterion in Lemma 2.[22] Substituting the solution for v from Lemma 1 then yields

$$\bar{X} = \frac{\omega}{\sqrt{2r}} + \frac{\sigma}{\sqrt{2r}} W_0\left(-\left(\frac{\omega}{\sigma} + 1\right) e^{-\frac{\omega}{\sigma}-1}\right) \tag{11}$$

as the investing threshold, where $W_0(\cdot)$ denotes the principal branch of the product logarithm.[23] One can readily see that (11) is consistent with the abandonment threshold in Lemma 1 in the sense that $\bar{X} > \underline{Y}$.[24] The initial value of any new investment that the firm makes is therefore

$$v(\bar{X}) = \frac{\bar{X}}{r} + \frac{\sigma}{\sqrt{2r^3}} e^{\frac{\sqrt{2r}}{\sigma}(\underline{Y}-\bar{X})} = \frac{\omega}{\sqrt{2r^3}}\left(1 - e^{-W_0\left(-\left(\frac{\omega}{\sigma}+1\right)e^{-\frac{\omega}{\sigma}-1}\right)-\frac{\omega}{\sigma}-1}\right) \tag{12}$$

Consistent with a classic result in real options theory, $v(\bar{X})$ is strictly positive, so that there exists some range of investment values $\left(0, v(\bar{X})\right)$ at which the firm forgoes the immediate implementation of profitable opportunities and instead chooses to wait.[25] The model by Zhang (2000), by comparison, features European option-style growth opportunities with fixed implementation

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[22] Logarithmic concavity is a weaker condition than concavity. Every concave function that is non-negative on its domain is also logarithmically concave, and so the model also admits a broad class of alternative specifications of v.
[23] The product logarithm W is defined as the solution to $x = W(x)e^{W(x)}$. The principal branch W_0 obtains after restricting the codomain to $W > -1$, so that $W_0 \in (-1,0)$ for $x < 0$. A notable implication is that $\bar{X}$ may be negative for large values of σ, i.e., the firm may find it optimal to implement initially loss-making investments if they have a sufficiently large abandonment option value.
[24] The counterintuitive outcome $\underline{Y} > \bar{X}$ can only obtain for liquidation values so large that the firm would prefer to liquidate its investments the instant it initiates them, which is not an economically sensible scenario.
[25] McDonald and Siegel (1986) derive this result for the case of a single investment, which corresponds to $\mu = \infty$. The forgoing of nominally profitable investments also appears consistent with firms' capital budgeting practices, which tend to apply a hurdle rate higher than firms' cost of capital.

dates, in which case delaying investment is not possible and the optimal strategy is to take any project with $v \geq 0$.[26]

Somewhat curiously, the optimal investing strategy is independent of the opportunity cost parameter μ. This invariance arises because an increase in μ has two countervailing effects. First, the drop in option value incurred upon investing is more severe for higher μ as the distance by which X needs to increase to reach $\bar{X}$ again is greater. Yet, a higher μ also means that, ceteris paribus, the option to invest is less valuable to begin with. The two effects cancel: an investment by a firm with high μ creates a large proportional decrease in an option value that was small to begin with, whereas an investment by a firm with low μ creates a small proportional decrease in an option with a high original value.[27] A non-trivial dependence of $\bar{X}$ on μ does, however, arise if generating investment opportunities is costly for the firm, a scenario explored in Section 3.3.

Total firm value m is now the sum of two components: the intrinsic values of the firm's investments in place, as previously defined in (6), and the growth option value identified in Lemma 2, or, formally,

$$m(\boldsymbol{Y}, X; t) = \sum_{i=1}^{n} v(Y_i)\mathbf{I}_{t \leq \bar{T}_i} + g(X) \tag{13}$$

Similarly, firm value as inferred based on the aggregate earnings figure P will be redefined as

$$\widehat{m}(P; t) = E(m(\boldsymbol{Y}, X; t)|P) = E\left(\sum_{i=1}^{n} E\left(v(Y_i)\mathbf{I}_{t \leq \bar{T}_i}\middle|n\right)\middle|P, t\right) + E(g(X)|P, t) \tag{14}$$

The object of interest continues to be the behavior of $\widehat{m}$ as a function of P.

Before turning attention to the more intricate case of the growth option component g, one should briefly revisit the properties of the firm's existing investments. The introduction of stochastic growth opportunities makes the creation of investments non-deterministic but leaves the relationship between earnings and the investment value v unchanged. Hence, although computing the value of investments in place, given P, now involves taking an expectation over all possible investment histories, each of these histories behaves as in the static setting treated in the

[26] A noteworthy subtlety is that $v(\bar{X})$ does not imply $\bar{X} > 0$. A negative investing threshold $\bar{X} < 0$ obtains for $\omega < \frac{\sigma}{e-1}$, i.e., it can be optimal to launch loss-making investments if high cash flow volatility makes the abandonment option sufficiently valuable.

[27] This rationale has an important subtlety: it is the investment value $v(\bar{X})$ that is invariant under μ, not necessarily the threshold $\bar{X}$ itself. To appreciate this point, suppose that the investment value were defined as $v(\bar{X} - \mu)$ instead of $v(\bar{X})$. Solving for the optimal threshold now yields a solution $\bar{X}$ that contains μ as an additive term, but the resulting investment profit at $\bar{X}$ would be the same as before.

previous section. The intuition for why investment value can decrease in earnings, as developed around Proposition 1, therefore remains intact, and one obtains the following generalization.

Corollary 1.1. *For sufficiently large* σ *or sufficiently small* ω*, the value of investments in place is decreasing in earnings around* $P = i\underline{Y}$*, for any* $i \in \mathbb{N}$.

The relationship between current earnings and future growth opportunities is less straightforward. In fact, casual inspection suggests that earnings P have no direct connection to the option value g at all. A supporting intuition is readily at hand: current earnings are the product of the firm's *past* investing activities and therefore have no immediate causal link to future investments. Two main arguments stand against this reasoning. First, the cash flows Y_i from existing investments and the evolution of the opportunity process X may be correlated for exogenous reasons. For example, an increase in sales of the current collection of a fashion company might indicate that the company's popularity among consumers has increased in general and that future collections can be expected to sell better as well. For the moment, the analysis will disregard the possibility of such mechanical, exogenously created correlation and continue with the assumption that the random increments in X are uncorrelated with the Y_i. The discussion will revisit the issue later and, for now, focus on a second link between P and g, which is an endogenous product of the firm's investing strategy.

The indifference condition in equation (10), which implicitly characterizes the firm's optimal investing strategy, holds the key to understanding why earnings and future investment are not independent even in the absence of exogenous correlation. Given the optimal investing rule identified in Lemma 2, the value of X predictably decreases from $\bar{X}$ to $\bar{X} - \mu$ each time the firm invests, while earnings simultaneously increase by $\bar{X}$. The synchronicity thus induced by past investment dates creates an endogenous link between P and X. To illustrate the consequences, consider a large sample of ex-ante identical firms that commence operations together at time T_0, and examine the relationship between P and X at some later time $t > T_0$. A reasonable inference is that those firms with earnings of greater magnitude at time t have, on average, made more investments. Hence, their most recent investment date T_n should generally be closer to time t than among firms with smaller earnings, whose average frequency of investment has been lower. The final investment date T_n, in turn, determines the distribution of X at time t because

$$\Pr(X(T_n) = \bar{X} - \mu | P, t) = \Pr(X(T_n) = \bar{X} - \mu) = 1$$

for all P and t, regardless of the value of T_n. While the actual dependence of T_n on P will turn out to be more nuanced than in the example sketched out above, the critical insight for now is that earnings relate to a firm's growth option value indirectly through the most recent investment date T_n.

It may seem difficult to envision how this rather technical and abstract argument relates to actual business practice. The analogy is, in fact, quite direct. Random events in the environment, akin to X, constantly change the value of a firm's business opportunities. When the firm decides to exploit one of these opportunities, the option value associated with the opportunity is necessarily extinguished, and a cash-flow producing investment is created instead. The pool of future opportunities thus mechanically shrinks whenever an investment occurs. At the same time, the firm's exercising its option to invest also implies that the opportunity pool must have had a sufficiently high value to begin with, or else the firm would have decided to wait, analogous to requiring X to reach $\bar{X}$ before taking action. Inferring growth option value from the most recent investment date therefore amounts to asking: how have opportunities most likely evolved in the meantime, given that the environment must have been favorable at the last investment date?

This train of thought leads to the problem of computing the option value g from T_n. Formally, the solution amounts to taking an expectation of $g(X)$ over the distribution of X at time t, conditional on T_n. In view of Lemma 2, g is increasing and convex in X, and the expected value of g at time t should therefore increase both in the mean of this conditional distribution and, by Jensen's inequality, also in the variance. Now, the further T_n lies in the past, the higher the conditional variance of X but the lower its conditional mean at time t, and thus a more recent T_n has two countervailing effects on the growth option value.[28] As the formal results below will demonstrate, the variance effect wins in the short term while the mean effect prevails on the long term. In other words, when the most recent investment date T_n is already close to today's date t, increasing T_n reduces the growth option value. When T_n lies further in the past, on the other hand, increasing T_n raises the option value.

[28] The process underlying X is Brownian motion with an upper boundary at $\bar{X}$. Conditioning the distribution of X at time t on T_n means conditioning on the event that X has not reached $\bar{X}$ between T_n and t. The variance of this distribution increases in $|t - T_n|$, and hence that X is close to $\bar{X}$ at time t without having hit the bound at least once in the meantime becomes increasingly unlikely as $|t - T_n|$ increases.

As an illustration of this non-monotonic relationship between T_n and g, consider a developer of mobile devices investing in new generations of its product over time. If the most recent product release occurred just last month, it is highly unlikely that the market has ripened for another new product already. While the recent product release indicates that market conditions are generally good, the company is probably not immediately going to reach the profit potential justifying a new investment. If the last product release instead occurred six months ago, there has been more time for events to occur that affect technology and market demand. Given the convexity of option values, these random events increase, in expectation, the value of the company's investment opportunities, so that the average growth option value is larger than it was just one month after the last investment. If, on the other hand, it has been several years since the last new product, it stands to reason that the firm has failed to keep up with the technological developments in the industry and is unable to develop competitive products anymore. The option value would be low in this case because the distance between the value of current opportunities and the level required to make investment worthwhile is, in expectation, very large, which now outweighs the variance effect.

With the link between T_n and g established, the question that remains is: what information does the earnings number P convey about T_n? The answer comes in two conceptually distinct parts. First, earnings grow by $\bar{X}$ each time a new investment is made, and hence firms with large earnings have, on average, made more investments than firms with small earnings. A higher frequency of investing in turn implies a smaller average distance $|t - T_n|$. Importantly, the logic applies identically to loss firms: in view of the previous discussion on the optimal abandonment strategy, negative earnings of large magnitude imply a greater number of surviving investments and thus, in expectation, also a greater number of investments made to begin with. The second way in which P relates to T_n has to do with the average investment's cash flow level. Each investment starts with a cash flow of $Y(T) = \bar{X}$ at its initiation date T. As time passes, the variance of the cash flow distribution increases, i.e., large deviations of Y from $\bar{X}$ become more likely. Thus, one would infer that an investment producing a cash flow close to $\bar{X}$ was most likely started only a short time ago, while a cash flow substantially above or below $\bar{X}$ suggests an investment date further in the past. The same rationale applies to multiple investments, so that an earnings figure of exactly $P = n\bar{X}$, for $n = 1,2, \ldots$, implies a shorter distance $|t - T_n|$ than a slightly higher or lower value of P.

As one might expect, the two interpretations of P interact. A change in earnings around $P = n\bar{X}$ can mean that the firm has n investments whose cash flows are each more or less close to $\bar{X}$, and one might accordingly draw an inference about T_n based on the cash flow level interpretation of P. On the other hand, an upward or downward movement away from $P = n\bar{X}$ also increases the likelihood that one is observing earnings generated by $n + 1$ or $n - 1$, rather than n, investments, which yields an inference in line with the investment frequency interpretation of P. A similar conclusion applies to firms with negative earning numbers around multiples of the abandonment threshold $\underline{Y}$. As P decreases toward $n\underline{Y}$, one may either infer that the firm has n investments that perform increasingly poorly, or that it has become more likely that the firm has $n + 1$ instead of n investments.

As the following results demonstrate, the cash flow level interpretation can dominate locally around multiples of $\bar{X}$ and $\underline{Y}$, while the investment frequency interpretation describes the global relationship between P and g, as one moves from low to high multiples of $\bar{X}$ and $\underline{Y}$. For ease presentation, the proposition is stated in two parts. Part 2a describes the growth option value as a function of positive earnings, and part 2b considers negative earnings. To facilitate the discussion, it will be convenient to define

$$\hat{g}(P; t) = E(g(X)|P, t)$$

as the growth option value given earnings of P, and

$$\hat{g}(P \in \mathbb{P}; t) = E(g(X)|P \in \mathbb{P}, t)$$

as the growth option value in expectation over some set of earnings values $\mathbb{P} \subset \mathbb{R}$.

Proposition 2a. *For sufficiently small σ, there exists a sequence of earnings levels $P_n < n\bar{X} < P_{n+1}$, for $n = 1,2, \ldots$, such that $\hat{g}(P \in (P_n, P_{n+1}); t)$ is quasiconcave in n, and, for every n, $\hat{g}(P; t)$ is quasiconcave in P on $(P_n, n\bar{X})$ and on $(n\bar{X}, P_{n+1})$.*

Proposition 2b. *For sufficiently small ω, $\hat{g}(P \in (n\underline{Y}, (n-1)\underline{Y}); t)$, for $n = 1,2, \ldots$, is quasiconcave in n. For sufficiently small ω and σ, there exists a sequence of earnings levels $P_n \in (n\underline{Y}, (n-1)\underline{Y})$ such that, for every n, $\hat{g}(P; t)$ is quasiconcave in P on $(P_n, (n-1)\underline{Y})$.*

As a visual aid in parsing these statements, Figure 2 shows a plot of growth option values and earnings, based on a simulation of the model. In essence, the proposition only describes in formal terms the outcomes of the logic discussed above. The first claim in part 2a states that, as earnings increase from about zero to large positive values, the growth option value, if computed as a kind of moving average over successive earnings intervals, rises initially but begins to decline again for high values of P, consistent with the investment frequency interpretation. In particular, since very large P imply that $|t - T_n| \approx 0$, one can readily see that the growth option value, as $P \to \infty$, converges to $g(\bar{X} - \mu)$. The second claim in part 2a makes a more nuanced, local statement by identifying intervals around integer multiples of $\bar{X}$ on which the growth option value rises and falls in line with the cash flow level interpretation of P. Graphically, this phenomenon appears in Figure 2 as a wave pattern in the plot, with local peaks around each $n\bar{X}$, for $n = 1,2, \ldots$.

Part 2b of the proposition makes two analogous claims about firms with negative earnings. First, the average growth option value between two neighboring multiples of the abandonment threshold $\underline{Y}$ increases initially as losses become larger but eventually declines again toward a limit value of $g(\bar{X} - \mu)$, consistent with the investment frequency interpretation of P.[29] The second, local claim is that there exist intervals between neighboring multiples of $\underline{Y}$ on which the growth option value changes in line with the intuition that, for a given number of investments, the average investment's cash flow moves away from $\underline{Y}$ and closer to $\bar{X}$ as P increases. As in the case of positive earnings, this phenomenon manifests itself by a wave pattern in the plot.

Proposition 2 holds for sufficiently small values of σ or ω. These qualifications are introduced to permit clean analytical statements about $\hat{g}$ but do not constitute necessary conditions. Simulations with a variety of parameter values suggest that the qualitative behavior of the model generally remains consistent with Proposition 2. A nonetheless noteworthy phenomenon is that part 2a turns from a statement about positive earnings into a statement about negative earnings when σ becomes larger. The reason is that, in view of Lemma 2, $\bar{X} < 0$ if $\sigma > \frac{\omega}{e-1}$, i.e., the firm optimally makes investments with negative initial cash flow if the cash flow variance, and thus the abandonment option value, are sufficiently high. Concerning the behavior of $\hat{g}$ for positive earnings in this case, the intuitions developed so far remain intact in the sense that larger positive earnings still imply a higher average frequency of investment.

[29] The decrease in growth option value as losses become very large does not appear in Figure __ because the simulation did not produce sufficiently large negative values of P.

There are three broad lessons from Proposition 2. First, part 2b reinforces the earlier finding that firm value can decrease in earnings among loss firms. Proposition 1 makes this observation with respect to investments in place and in the context of abandonment options, while Proposition 2 covers the future growth option component of firm value. In conjunction, these results suggest that optionality may have a role to play in explaining the empirical regularity of a negative association between earnings and firm value when earnings are negative. For visual illustration, Figure 3 shows a plot of earnings and total firm value based on a simulation of the model.

The second lesson from Proposition 2 is that the growth option value increases only slowly when earnings are high and even declines when firms show very high profits, as high profits are the outcome of firms' having drawn heavily from their pool of opportunities and thereby, on average, leaving fewer profitable investments for the future.[30] The result thus suggests that the value of growth options need not be monotonically increasing or convex in earnings, even in the absence of losses.[31] By comparison, Zhang (2000) obtains a monotonic and convex relationship between earnings and the growth option value in a model featuring a growth opportunity with a fixed future exercise date, analogous to a European call option. The combination of option style and the absence of a decision history account for this difference in option value behavior.[32]

The third lesson from Proposition 2 is that, locally, the change in option value in response to a small change in earnings is difficult to predict. Even though Figure 2 shows clear directional patterns if one considers large earnings changes, the local 'waviness' of the plot makes a cautionary statement that one cannot easily infer even the direction of the change in growth opportunities from a small increment in earnings. An exacerbating factor is that firms' investment opportunities in practice very likely do not follow the simple, time-consistent mechanics in this model.

At this point, it may be instructive to revisit the problem of correlation. The model assumes that the random increments in the state variable X are uncorrelated with the cash flows Y_i. Yet, the firm's endogenously determined decision logic creates a connection between the growth op-

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[30] Moreover, even if one were to condition on the number of investments in place, the cash flow level interpretation of P would still imply that $\hat{g}$ decreases at sufficiently high P.

[31] The empirical work by Freeman and Tse (1992) provides some related evidence. The authors test a mapping from unexpected earnings to unexpected returns with non-monotonic, S-shaped curvature, based on the conjecture that unexpected earnings of large magnitude are more likely to be transient and hence affect firm value less.

[32] The dynamic, capacity-based growth option model by Livdan and Nezlobin (2016) permits an interaction between cash flows and growth opportunities that has some similarity to the setup underlying Proposition 2, but the paper does not include an explicit result on the properties of the expected option value.

tion value (a function only of X) and the earnings (which depend only on the Y_i). Thus, Proposition 2 is, in fact, a statement about correlation between X and the Y_i, albeit not one imposed by environmental factors but one arising endogenously from an optimized decision strategy.

Superimposing a positive exogenous correlation between X and the Y_i on the existing setup would, on the surface, create some consistency with models in which the payoffs of investments in place act as the underlying of future growth opportunities.[33] With X and Y_i correlated positively, one might conjecture, the mapping from P to $\hat{g}$ should bear closer resemblance to a globally convex and monotonic relationship. The failure point of this reasoning lies precisely in the decision logic that creates the phenomena documented in Propositions 1 and 2. A joint increase in X and P that leads X to hit $\bar{X}$ would immediately trigger a new investment and thus both a concurrent reduction in the option value to $g(\bar{X} - \mu)$ and an additional increase in P by $\bar{X}$. Positive correlation would only increase the likelihood of this outcome and thus enhance the very mechanism that creates the results in Proposition 2 to begin with.[34] A second reinforcing effect arises for loss firms. A joint decrease in X and Y_i that leads Y_i to hit the abandonment threshold $\underline{Y}$ would lead to an increase in earnings while the option value declines. Correlation would therefore strengthen the inverse relationship between P and $\hat{g}$ when earnings are negative. The reason why the model yields a non-monotonic, non-convex relationship between earnings and firm value does therefore not lie in the absence of correlation between earnings and opportunities but in the simultaneity of earnings changes and option value changes.

The current level of the state variable X is thus not exogenous in the sense that it does not solely reflect adverse or favorable investing opportunities. Information about favorable developments could correspond to an increase in X if they imply that a business opportunity has gained in value, but the same information could also correspond to a decrease in X if the firm has already taken advantage of the opportunity. In other words, X is indicative not only of whether the firm's business environment is favorable or unfavorable but also of whether and how the firm

[33] For example, Zhang (2000) models the growth option as an opportunity to expand the size of the firm's operations at the same profitability as the firm's existing investments.

[34] Generally, correlation between current profitability and future investment opportunities could be either positive or negative. The former case might arise if high demand for the firm's current products indicates increased demand for the firm's products in general (including products to be developed in the future), while the latter case might arise if investing in new products when sales of existing products are high would lead to severe cannibalization effects. In addition, the correlation is likely strong only among recent investments. It is therefore not clear that non-zero correlation necessarily has a first-order effect, especially in firms with a long investment history.

has reacted to this environment. What type of information should correspond to high or low X in practice is therefore inherently unclear. In order to infer X from sources other than earnings, one needs to know not only whether, say, market demand has grown or a technological invention has been made, but also how many of the potential ways to benefit from this development the firm has already implemented. Compounding the problem, the two phenomena are correlated: as the value of the investment opportunity set grows, the probability that the firm begins to harvest these opportunities grows as well.

As a final caveat, one should note that the results obtained so far condition on t, the period of time over which investments have been made and earnings have accumulated. Hence, the results do not permit a direct comparison between firms with different t even if these firms face identical cash flow and opportunity processes. In practice, differences in t likely correspond to differences in firm age. Ceteris paribus, the older a firm is, the more time it has had to make investments that contribute to its current earnings, and hence two firms with the same amount of earnings may have different option values if they differ in age. In a cross-sectional empirical analysis of the relationship between earnings and firm value, it may therefore be important to consider firm age as a covariate in the research design.[35]

3.3. Costly Investment Opportunity Development and Shut-Down Options

As developed so far, the model does not provide an explanation for business failure. While the firm might decide to abandon any (and possibly all) of its existing investments, there is no reason to dissolve the business as a whole as long as the process X continues to generate future investment opportunities. Yet, business liquidations are commonplace in reality, suggesting that generating future investment opportunities and maintaining the organization's ability to take advantage of them comes at a cost that, under certain circumstances, might outweigh the benefits. This section examins the implications of this costly investment opportunity development.

The setup is the same as in the original model, except that the firm now additionally incurs a cost to maintain the opportunity-generating process. The expenditure that maintains X will be referred to as the firm's business development cost and takes the form of a continuous, constant

[35] A possible approach may be to estimate expected option values via changes in earnings between periods rather than through earnings levels.

cash outflow κ.[36] Real-word analogues of κ include expenditures for employee training, research and development, and marketing. Conceptually, the evolution of X thus represents both unexpected changes in the firm's environment, such as technological innovation, fashion or new legislation, and the firm's ability to benefit from these changes, e.g., by having the appropriate technology, brand recognition, or distribution network in place.[37]

As long as the firm expends κ, the opportunity process continues to evolve according to (3). If the firm ever decides to stop expending the development cost, the process terminates and the firm goes out of business. The termination decision is irrevocable, i.e., once the firm has shut down, there is no option to restart the process.[38] The real-world analogue to this situation is the dissolution of organizational structures when a firm enters liquidation: skilled employees leave, supplier networks break down, and customer contacts are lost.[39] When the firm shuts down, all investments in place are liquidated at their current intrinsic values.[40]

It stands to reason that the firm's optimal shut-down strategy is similar to the optimal abandonment rule for individual investments. For any given anticipated future liquidation strategy, the expected payoff to the firm is increasing in the current value of X (because the investment value increases in X). Hence, if liquidation is optimal for $X = \underline{X}$, liquidation should also be optimal for any $X < \underline{X}$, and so one should expect the firm's optimal liquidation strategy to be given by a threshold value $\underline{X}$. In other words, pricing the firm's future investing opportunities requires consideration of both put (shut-down) and call (growth) option features and thus, in formal terms, amounts to a hybrid result of Lemmas 1 and 2.

The preceding rationale is based on two conjectures: first, that the option exercise strategy is indeed given by threshold values $\underline{X}$ and $\bar{X}$, and, second, that the firm expects to continue operating after each new investment. The second conjecture is valid as long as $\bar{X} - \mu \geq \underline{X}$, i.e., the opportunity cost of investment is not so high that the firm would prefer to shut down immediately

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[36] Assuming a fixed (rather than stochastic) development cost is convenient but not essential because the process of interest is the one describing how the difference between costs and profits evolves, not either one in isolation.
[37] One can thus think of κ as generating intangible assets, such as technological knowhow, reputation, organizational culture, or customer and supplier relationships.
[38] The qualitative behavior of the model is not substantively affected if costly resumption of operations is permitted, but tractability is reduced. For a model involving the option to resume business at a cost, see Dixit (1989).
[39] This is not to say the firm's investors will never have the opportunity to start an enterprise again. Rather, the assumption implicit in this setup is that the possibility of starting a new firm is independent of whether the firm in this model is in existence or not.
[40] This assumption is natural since liquidating investments in place at a value other than v would imply that κ, contrary to its definition, also has a role in maintaining these investments.

after its first investment. The condition $\bar{X} - \mu \geq \underline{X}$ will therefore be referred to as the firm's going concern constraint. To avoid analyzing trivial scenarios, the firm is assumed to operate as a going concern in this section.[41] The following lemma verifies both conjectures formally and solves the program for the option value g.

Lemma 3. *If the firm operates as a going concern and the investment value $v(X)$ is logarithmically concave, the firm optimally invests at time $T_i = \inf\{s: X(s) \geq \bar{X}, s > T_{i-1}\}$, where $\bar{X}$ is the unique solution to*

$$v(\bar{X}) = \sqrt{\frac{\omega^2}{2r}\left(v'(\bar{X})\right)^2 - \frac{\kappa^2}{r^2} e^{\frac{\sqrt{2r}}{\omega}\mu}\left(1 - e^{-\frac{\sqrt{2r}}{\omega}\mu}\right)^2}$$

and the firm optimally shuts down at time $\bar{T} = \inf\{s: X(s) \leq \underline{X}\}$, where

$$\underline{X} = \bar{X} - \mu - \frac{\omega}{\sqrt{2r}} \ln \frac{rv(\bar{X}) + \frac{r\omega}{\sqrt{2r}} v'(\bar{X})}{\left(e^{\frac{\sqrt{2r}}{\omega}\mu} - 1\right)\kappa}$$

The associated growth option value is

$$g(X) = \frac{\kappa}{2r}\left(e^{\frac{\sqrt{2r}}{\omega}(X - \underline{X})} + e^{\frac{\sqrt{2r}}{\omega}(\underline{X} - X)} - 2\right)$$

The firm operates as a going concern if and only if $v(\bar{X}) \geq g(\underline{X} + \mu)$.

Lemma 3 subsumes the original setup underlying the growth option value in Lemma 2 as the special case $\kappa = 0$. An immediate consequence of setting $\kappa > 0$ is that $\bar{X}$ decreases, i.e., the firm undertakes investments with lower net present value than it does when $\kappa = 0$. The logic behind this outcome is that κ increases the cost of waiting for a better investment opportunity. Yet, Lemma 3 does not imply that one should generally expect firms with high research and development spending to make less profitable investments than firms that can generate investment opportunities at little or no cost. The reason is that raising κ tightens the firm's going concern constraint $\bar{X} - \mu \geq \underline{X}$, which, by Lemma 3, is equivalent to

$$v(\bar{X}) \geq g(\underline{X} + \mu) = \frac{\kappa}{2r}\left(e^{\frac{\sqrt{2r}}{\omega}\mu} + e^{-\frac{\sqrt{2r}}{\omega}\mu} - 2\right) \tag{15}$$

[41] Analyzing the one-shot scenario $\bar{X} - \mu < \underline{X}$ would be straightforward but not particularly useful given the objective of this paper, which is to explore intertemporal tradeoffs in the exercise of real options through time.

The left-hand side of (15) is monotonically decreasing in κ while the right-hand side is increasing, and so, for any given set of parameters σ, ω, r and μ, there exists a maximum development cost $\bar{\kappa}$ above which no firm would stay in business. Thus, firms with high κ should tend to require a combination of lower investment opportunity costs (lower μ), higher volatility in investment opportunities (higher ω) and a lower discount rate r in order for staying in business to be profitable, all of which increase $\bar{X}$ and hence investment profitability. The following corollary to Proposition 2 sheds some light on how the presence of the development cost and the ensuing firm-level liquidation option affects the mapping from earnings to the growth option value vis-à-vis the original setting with $\kappa = 0$.

Corollary 2.1. *For sufficiently small σ, there exists a sequence of earnings levels $P_n < n\bar{X} < P_{n+1}$, for $n = 1,2, \dots$, such that, for any $\kappa \in (0, \bar{\kappa})$, $\hat{g}(P \in (P_n, P_{n+1}); t)$ is decreasing in n when n is sufficiently large, and that $\hat{g}(P \in (P_n, P_{n+1}); t)$ is quasiconcave in n when $\kappa = 0$ and monotonically decreasing in n when $\kappa = \bar{\kappa}$.*

Proposition 2a has established that, among firms with high earnings, the growth option value, when averaged over successive earnings intervals, can decrease in earnings. The first claim in Corollary 2.1 extends this part of the result, which in Proposition 2a obtains for $\kappa = 0$, to strictly positive values of κ and thereby shows that the phenomenon is not an artifact of setting the development cost to zero. The rationale remains the same: high earnings indicate that the firm has likely taken advantage of an investment opportunity recently, which depletes the pool of future investment opportunities.

The more novel aspect lies in the final part of the corollary. The claim about the case $\kappa = 0$ of course merely restates the result in Proposition 2a: absent any business development cost, the growth option value is quasiconcave. By contrast, as κ reaches its theoretical maximum value $\bar{\kappa}$, the local result becomes global, i.e., the growth option value is no longer decreasing in earnings for high earnings only but becomes a monotonically decreasing function of earnings everywhere. To develop some intuition why development costs affect the result, recall that a firm with development costs at the maximum level $\kappa = \bar{\kappa}$ is, after each investment, just about indifferent between continuing operations and terminating the business and thus has an option value of zero at that time. The option value therefore appreciates in expectation the longer the firm continues without making an investment because the absence of a liquidation event indicates, in expectation, an upward movement in X. If $\kappa = 0$, on the other hand, X is not bounded below and thus the absence of new investments induces increasingly unfavorable beliefs about X as time passes.

As a real-world illustration of the case $\kappa \approx \bar{\kappa}$, consider a biotechnology company whose research efforts are entirely concentrated on the development of one particular medical drug. No earnings are generated while the drug is under development, but the possibility of a success implies a substantial option value. In fact, as long as the company is working on the development, the expected option value rises over time even though no earnings are generated, because the mere fact that the company has not given up and shut down is evidence of favorable developments. On the other hand, once the development has succeeded, earnings are generated but the option is extinguished, and since the company has no further products under development at that time, the option value associated with future opportunities is small. In other words, each investment undertaken increases earnings but causes a severe drop in option value. Corollary 2.1 is thus consistent with the phenomenon that innovative, technology-intensive companies often experience substantial appreciation in market value over time even though they may generate no earnings yet.

4. Conclusion

This paper studies the relationship between earnings and firm value in a dynamic real options setting. The model yields a theoretical basis for the empirically documented phenomenon that firm value can decrease in earnings among loss-making firms. Results further show that the value of future growth opportunities can be a multimodal function of earnings and, in particular, begin to decline when earnings are very high. The latter phenomenon becomes particularly pronounced if firms face a high cost of developing new investment opportunities. More broadly, the model suggests that, even with a highly parsimonious parameterization of the problem, the curvature of the earnings-value map has a non-trivial form and cannot generally be characterized in elementary terms.

This rather nuanced conclusion differs from the typically convex and monotonic relationship between earnings and value obtained in static option models, which price a fixed endowment of future opportunities whose underlying is the firm's current earnings. The reason for the difference does not lie in any esoteric modeling choices. The paper's results rely on only two fundamental assumptions: first, that firms have discretion over when to initiate and abandon investment projects, and, second, that firms exercise this discretion in order to maximize value creation. The model does, however, permit the exogenous arrival of opportunities and firms' endogenously determined reaction to them to occur simultaneously and continuously through time. As a result, earnings can no longer be viewed as an underlying to the firm's options but rather act as a covariate of firm value, as past investing and abandonment decisions jointly determine the firm's current earnings as well as the value of its future opportunities. In particular, past investing and abandoning decisions increase current earnings while at the same time both reducing the set of opportunities left for the future as well as changing the number of investments in place.

The more general implication of this dynamic, and arguably not unrealistic, view of firms' operations is that valuation, whether based on earnings or other contemporaneous performance variables, becomes a 'coarse' computation in the sense that both value and current performance are simultaneously determined by a potentially large and unobservable number of interactions between exogenous, stochastic opportunities and endogenously optimized decisions taken in response. An inference about firm value from earnings therefore amounts to relying on an endogenously created correlation between the two rather than on an immediate causal link. What-if analyses asking by how much an increase in earnings would change firm value are meaningless in this context. Current earnings are the product of past decisions made in response to past opportunities that, because they have now been acted upon, have either changed in value or disappeared altogether.

Despite these limitations, the results may prove useful in several respects. First, as a practical matter, valuation analysis must rely on limited, observable performance data, with earnings often occupying center stage. Understanding how earnings relate to value is therefore critical even if this relationship takes the form of a non-trivial and descriptive correlation rather than immediate causality expressed in closed form. Second, the earnings-value relationship has been studied extensively in prior research and has shaped and created perceptions of how accounting informs investors (Kothari 2001; Dechow, Ge and Schrand 2009). Hence, even if the problems raised by the model were a non-issue to practitioners, revisiting the earnings-value relationship in a real-options setting can add context to existing research results.

The model is deliberately structured as parsimoniously as possible. In particular, earnings equate to the aggregation of investment-level cash flows that follow a simple, symmetric stochastic process consistent with the Feltham-Ohlson model. The simplicity of the parameterization does not constitute a necessary condition for any results. Rather, it serves to emphasize that assuming more complex properties of earnings, such an asymmetric recognition of gains and losses, is not required in order to explain much of the seemingly irregular behavior of the earnings-value relationship documented in empirical work. At the same time, the model can serve as a benchmark against which more complex properties of accounting information can be tested empirically, and as a foundation for theoretical models of these properties.

Appendix

Proof of Lemma 1. The distribution of Y_i does not depend on time t, and hence the optimal abandonment strategy does not depend on t. Further, $Y_i(t)$ is a Markov process, and so the abandonment decision depends only on the current value of Y and not on its history. The proof begins with the conjecture that the optimal abandonment rule takes the form of a unique threshold $\underline{Y}$ and proceeds to verify whether the following optimality conditions yield a unique solution. For ease of notation, subscripts i are omitted. Three conditions characterize the solution. First, the expected return on the investment value must equal the discount rate r, and thus, by Itō's lemma,

$$\frac{\sigma}{2}v''(Y) + Y = rv(Y) \tag{A1}$$

at all $Y \geq \underline{Y}$. At the termination threshold $\underline{Y}$, the firm must be indifferent between continuing and abandoning the investment, which requires the value-matching condition

$$v(\underline{Y}) = 0 \tag{A2}$$

where the right-hand side is the net liquidation value, set to zero in this model. Finally, the termination threshold $\underline{Y}$ must be optimal in the sense that there may not exist any upward or downward adjustment to $\underline{Y}$ that increases v, which requires the smooth-pasting condition

$$v'(\underline{Y}) = 0 \tag{A3}$$

The general solution to (A1) is

$$v(Y) = \frac{Y}{r} + Ae^{\frac{\sqrt{2r}}{\sigma}Y} + Be^{-\frac{\sqrt{2r}}{\sigma}Y}$$

where A and B are unknown constants. As $Y \to \infty$, the value of the abandonment option must vanish and the investment value must approach the perpetuity of Y, which implies that $A = 0$. Then (A2) and (A3) become

$$\frac{\underline{Y}}{r} + Be^{-\frac{\sqrt{2r}}{\sigma}\underline{Y}} = 0 \tag{A4}$$

and

$$\frac{1}{r} - \frac{\sqrt{2r}}{\sigma}Be^{-\frac{\sqrt{2r}}{\sigma}\underline{Y}} = 0 \tag{A5}$$

respectively. The unique solution to (A4) and (A5) is

$$\underline{Y} = -\frac{\sigma}{\sqrt{2r}}$$

and

$$B = \frac{\sigma}{\sqrt{2r^3}} e^{-1}$$

which yields the claimed result.

Proof of Proposition 1. Consider earnings of $P = i\underline{Y} - \frac{\varepsilon}{2}$, where $i = 1, \dots, n-1$ and $\varepsilon \in (0, -2\underline{Y})$. The aim is to show that, for any i, there exist t and ε (σ and ε) such that

$$\widehat{m}(P,t) > \widehat{m}(P+\varepsilon,t) \tag{A6}$$

In view of Jensen's inequality and the convexity of the intrinsic investment value v, the left-hand side of (A6) can be bounded below by

$$\widehat{m}(P,t) \geq v\left(\frac{n-1}{n}\underline{Y} - \frac{\varepsilon}{2n}\right)$$

For the left-hand side, observe first that

$$\Pr\left(n^*(t) < i \middle| P = i\underline{Y} + \tfrac{\varepsilon}{2}\right) = 0$$

where

$$n^*(t) = \sum_{j=1}^{n} \mathbf{I}_{t \leq \bar{T}_j}$$

is the number of active investments as of time t. The reason is that $n^*(t) < i$ would imply that there exists at least one investment with a cash flow below $\underline{Y}$, which would be inconsistent with the firm's optimal abandonment strategy identified in Lemma 1. The right-hand side can thus be bounded above by

$$\begin{aligned}
\widehat{m}(P+\varepsilon,t) &= \sum_{j=i}^{n} \Pr\left(n^*(t) = j \middle| P = i\underline{Y} + \tfrac{\varepsilon}{2}\right) \widehat{m}(P,t|n^*(t) = j) \\
&< \Pr\left(n^*(t) = i \middle| P = i\underline{Y} + \tfrac{\varepsilon}{2}\right) v\left(\underline{Y} + \frac{\varepsilon}{2}\right) \\
&+ \left(1 - \Pr\left(n^*(t) = i \middle| P = i\underline{Y} + \tfrac{\varepsilon}{2}\right)\right) v(-n\underline{Y})
\end{aligned} \tag{A7}$$

where the final inequality again follows from the convexity of v. Then the inequality in (A7) holds for large t and ε close to zero if it can be shown that

$$\lim_{t \to \infty} \Pr\left(n^*(t) = i \middle| P = i\underline{Y} + \tfrac{\varepsilon}{2}\right) = 1$$

for any ε. To this end, it suffices to demonstrate that the likelihood ratio of $n^*(t) = i$ to $n^*(t) = j$, for any $j > i$, diverges at $t \to \infty$. For the moment, let $T_i = 0$ for all $i = 1, \dots, n$. Then the probability density that project i survives through time t and generates a payoff of y_i takes the well-known form

$$f(y_i; t) = \frac{\phi\left(\frac{y_i - \bar{Y}}{\sigma\sqrt{t}}\right) - \phi\left(\frac{y_i - 2\underline{Y} + \bar{Y}}{\sigma\sqrt{t}}\right)}{\sigma\sqrt{t}}$$

where $\bar{Y}$ is the initial cash flow at time T_i and $\phi(\cdot)$ is the density function of a standard normal distribution (Ross 1996). By complementarity, the probability that a given investment is abandoned before time t is

$$\bar{F}(t) = 1 - F(t) = 1 - \int_{\underline{Y}}^{\infty} f(y; t)\, dy = 2\Phi\left(\frac{\underline{Y} - \bar{Y}}{\sigma\sqrt{t}}\right)$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal distribution. Then the probability that earnings P are generated by i active investments at time t is

$$\Pr(n^*(t) = i | P) = \int_{\underline{Y}}^{P-(i-1)\underline{Y}} \dots \int_{\underline{Y}}^{P-\underline{Y}-y_1-\cdots y_{i-2}} \prod_{j=1}^{i-1} f\left(y_j; t\right) f(P - y_1 - \cdots y_{i-1}; t) \bar{F}^{n-i}(t)\, d\boldsymbol{y}$$

Note that, as long as $\varepsilon > 0$, $\Pr(n(t) = i|P) > 0$ for all i and $P > i\underline{Y}$. Consider now the likelihood ratio of $\Pr(n(t) = i|P)$ and $\Pr(n(t) = i - 1|P)$. The integrand ratio of any two vectors of cash flows $\boldsymbol{y} = (y_1, \dots, y_i)$ for $n^*(t) = i$ and $\boldsymbol{z} = (z_1, \dots, z_{i-1})$ for $n^*(t) = i - 1$ is

$$\frac{\prod_{j=1}^{i} f\left(y_j; t\right) f(P - y_1 - \cdots y_i; t) \bar{F}^{n-i-1}(t)}{\prod_{j=1}^{i-1} f\left(z_j; t\right) f(P - z_1 - \cdots z_{i-1}; t) \bar{F}^{n-i}(t)} = \frac{\prod_{j=1}^{i} f\left(y_j; t\right) f(P - y_1 - \cdots y_i; t)}{\prod_{j=1}^{i-1} f\left(z_j; t\right) f(P - z_1 - \cdots z_{i-1}; t) \bar{F}(t)}$$

Then, at any set of cash flows with positive densities,

$$\lim_{t\to\infty} \frac{\prod_{k=1}^{i} f\left(y_j; t\right) f(P - y_1 - \cdots y_i; t)}{\prod_{j=1}^{i-1} f\left(z_j; t\right) f(P - z_1 - \cdots z_{i-1}; t) \bar{F}(t)}$$

$$= \prod_{j=1}^{i-1} \frac{\left(y_j - \underline{Y}\right)(P - y_1 - \cdots - y_{i-1} - \underline{Y})}{\left(z_j - \underline{Y}\right)(P - z_1 - \cdots - z_{i-1} - \underline{Y})} \lim_{t\to\infty} \frac{f(y_i; t)}{\bar{F}(t)}$$

which converges to zero in view of

$$\lim_{t\to\infty} \bar{F}(t) = 1$$

and

$$\lim_{t\to\infty} f(y; t) = 0$$

at all y, and because

$$\lim_{t\to\infty}\frac{f(y_j;t)}{f(z_j;t)}=\lim_{t\to\infty}\frac{\phi\left(\frac{y_j-\bar{Y}}{\sigma\sqrt{t}}\right)-\phi\left(\frac{y_j-2\underline{Y}+\bar{Y}}{\sigma\sqrt{t}}\right)}{\phi\left(\frac{z_j-\bar{Y}}{\sigma\sqrt{t}}\right)-\phi\left(\frac{z_j-2\underline{Y}+\bar{Y}}{\sigma\sqrt{t}}\right)}=\lim_{t\to\infty}\frac{1-e^{\frac{(y_j-\bar{Y})^2-(y_j-2\underline{Y}+\bar{Y})^2}{\sigma^2 t}}}{1-e^{\frac{(z_j-\bar{Y})^2-(z_j-2\underline{Y}+\bar{Y})^2}{\sigma^2 t}}}$$

$$=\frac{(y_j-\bar{Y})^2-(y_j-2\underline{Y}+\bar{Y})^2}{(z_j-\bar{Y})^2-(z_j-2\underline{Y}+\bar{Y})^2}=\frac{y_j-\underline{Y}}{z_j-\underline{Y}}$$

which is finite. Hence,

$$\lim_{t\to\infty}\frac{\Pr(n(t)=i|P)}{\Pr(n(t)=i-1|P)}=0$$

for any i, and, a fortiori, also for any $j<i-1$. Then $\Pr(n(t)=i|P)\to 1$ as $t\to\infty$, and so, for any ε, there exists t sufficiently high that

$$\widehat{m}(P+\varepsilon,t)\le v\left(\underline{Y}+\frac{\varepsilon}{2}\right)+\varepsilon$$

Setting ε sufficiently close to zero yields (A6). To apply the argument to σ instead of t, it suffices to observe that $f(y;t)$ vanishes at any finite y as $\sigma\to\infty$. To generalize the result to $T_i\neq 0$, one can replace t in f and $\bar{F}$ with $t-T_i$, which does not affect the argument.

Proof of Lemma 2. By the same argument as in Lemma 1, the option value only depends on the current value of X. Conjecturing that the optimal investing strategy is given by a threshold value $\bar{X}$ yields the program

$$\frac{\omega^2}{2}g''(X)=rg(X) \tag{A8}$$

at all X,

$$g(\bar{X})=g(\bar{X}-\mu)+v(\bar{X}) \tag{A9}$$

as the value-matching condition, and

$$g'(\bar{X})=g'(\bar{X}-\mu)+v'(\bar{X}) \tag{A10}$$

as the smooth-pasting condition. It remains to be verified that the program indeed yields a unique solution. The general solution to (A8) is

$$g(X)=Ae^{\frac{\sqrt{2r}}{\omega}X}+Be^{-\frac{\sqrt{2r}}{\omega}X}$$

where A and B are unknown constants. The option value must vanish as $X \to -\infty$, and hence $B = 0$. Then (A9) and (A10) become

$$Ae^{\frac{\sqrt{2r}}{\omega}\bar{X}} = Ae^{\frac{\sqrt{2r}}{\omega}(\bar{X}-\mu)} + v(\bar{X}) \tag{A11}$$

and

$$A\frac{\sqrt{2r}}{\omega}e^{\frac{\sqrt{2r}}{\omega}\bar{X}} = A\frac{\sqrt{2r}}{\omega}e^{\frac{\sqrt{2r}}{\omega}(\bar{X}-\mu)} + v'(\bar{X}) \tag{A12}$$

respectively. Solving (A11) and (A12) yields

$$A = \frac{e^{-\frac{\sqrt{2r}}{\omega}\bar{X}}}{1 - e^{-\frac{\sqrt{2r}}{\omega}\mu}} v(\bar{X})$$

and

$$\frac{v'(\bar{X})}{v(\bar{X})} = \frac{\sqrt{2r}}{\omega}$$

which indeed have a unique solution if $v(X)$ is logarithmically concave at all $v(X) > 0$.

Proof of Corollary 1.1. The claim for low σ follows from Proposition 1, as the likelihood ratio result applies to all investment histories regardless of their prior probabilities. For the claim about ω, note first that, given $X = \bar{X} - \mu$ at each investment date, the probability density of an investment at time t takes the well-known form

$$\eta(t - T_n) = \frac{\mu}{\omega\sqrt{(t - T_n)^3}}\phi\left(\frac{-\mu}{\omega\sqrt{t - T_n}}\right)$$

of an inverse Gaussian distribution, where T_n is the date of the most recent investment (Ross 1996). The joint probability that a given earnings value P is generated by cash flows $\boldsymbol{y} = (y_1, \dots, y_n)$ and an investment history $\boldsymbol{T} = (T_1, \dots, T_n)$ is thus

$$\Pr(\boldsymbol{y}, \boldsymbol{T}; t) = \prod_{i=1}^{n} f(y_i; t - T_i)\eta(T_i - T_{i-1})H(t - T_n)$$

subject to the constraint that the y_i add to P, where

$$H(t - T_n) = 1 - \int_0^t \eta(s)\, ds = 1 - 2\Phi\left(\frac{-\mu}{\omega\sqrt{t - T_n}}\right)$$

is the probability that no further investment occurred between T_n and t, and

$$f(y;t) = \frac{\phi\left(\frac{y-\bar{X}}{\sigma\sqrt{t}}\right) - \phi\left(\frac{y-2\underline{Y}+\bar{X}}{\sigma\sqrt{t}}\right)}{\sigma\sqrt{t}}$$

One should note that $H \to 1$ as $\omega \to 0$. If any investment i has been abandoned by time t, f is replaced by the abandonment probability for this i. As $\omega \to 0$, the likelihood ratio of earnings P generated by n investments to earnings P generated by $n+1$ investments, with identical investment dates T_1 through T_n and arbitrary T_{n+1}, becomes

$$\lim_{\omega\to 0} \frac{\prod_{i=1}^{n} f(y_i; t-T_i)\eta(T_i - T_{i-1})H(t-T_n)}{\prod_{i=1}^{n+1} f(z_i; t-T_i)\eta(T_i - T_{i-1})H(t-T_{n+1})} = \frac{\prod_{i=1}^{n} f(y_i; t-T_i)}{\prod_{i=1}^{n+1} f(z_i; t-T_i)} \lim_{\omega\to 0} \frac{1}{\eta(T_{n+1}-T_n)} = \infty$$

for any arbitrary cash flows $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{z} = (z_1, \dots, z_{n+1})$ that each sum to P. This conclusion holds a fortiori for all $n+i$, with $i = 2,3,\dots$. At the same time, P can only be generated by n or fewer investments if $P > n\underline{Y}$, in view of the optimal abandonment policy. It follows that

$$\lim_{\omega\to 0} \Pr(n = i | P \in (i\underline{Y}, (i-1)\underline{Y}), t) = 1$$

for all $i = 1,2,\dots$, where all n investments must have survived through time t. One can from hereon replicate the argument in Proposition 1 that there exists $\varepsilon > 0$ such that

$$\widehat{m}(n\underline{Y} - \varepsilon; t) > \widehat{m}(n\underline{Y} + \varepsilon; t)$$

for sufficiently small ε.

Proof of Proposition 2a. It will first be shown that the growth option value, conditional on the most recent investment date T_n, is quasiconcave in T_n. This result is extended to expected growth option values conditional on n or P. To this end, define

$$\hat{g}(\cdot; t) \equiv E(g(X)| \cdot, t)$$

Then the growth option value, conditional on earnings, can be written as

$$\hat{g}(P;t) = \sum_{n=1}^{\infty} \hat{g}(n;t) \Pr(n;t|P) = \sum_{n=1}^{\infty} \int_0^t \hat{g}(T_n;t) \Pr(T_n;t|n) \Pr(n;t|P)\, dT_n$$

It will first be established that $\hat{g}(T_n;t)$ has a unique interior maximum in T_n. Given an investing strategy with threshold $\bar{X}$ and the date T_n of the most recent investment, the probability density of $X(t)$ at time t is

$$h(x; t-T_n) = \frac{\phi\left(\frac{x-\bar{X}+\mu}{\omega\sqrt{t-T_n}}\right) - \phi\left(\frac{x-\bar{X}-\mu}{\omega\sqrt{t-T_n}}\right)}{\omega\sqrt{t-T_n}} \qquad \text{(A13)}$$

where $\phi(\cdot)$ is the density function of the standard normal distribution (Ross 1996). Define $\tau \equiv \omega\sqrt{t - T_n}$ and $\xi \equiv \frac{\sqrt{2r}}{\omega}$. In view of Lemma 2, the expected investment option value at time t can then equivalently be written as a function of τ in the form of

$$\hat{g}(T_n; t) = \hat{g}(\tau) \propto E\left(e^{\frac{\sqrt{2r}}{\omega}X}\middle|\tau\right) = \int_{-\infty}^{\bar{X}} e^{\xi x}\frac{h(x;\tau)}{H(\tau)}dx$$
$$= e^{\xi\bar{X}+\frac{\xi^2\tau^2}{2}}\frac{e^{-\xi\mu}\Phi\left(-\xi\tau+\frac{\mu}{\tau}\right)-e^{\xi\mu}\Phi\left(-\xi\tau-\frac{\mu}{\tau}\right)}{1-2\Phi\left(-\frac{\mu}{\tau}\right)} \tag{A14}$$

where

$$H(\tau) = \int_{-\infty}^{\bar{X}} h(x;\tau)\,dx = 1 - 2\Phi\left(-\frac{\mu}{\tau}\right) \tag{A15}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. The expected investment option value reaches a stationary point when

$$\hat{g}'(\tau) \propto \xi^2\tau\hat{g}(\tau) + e^{\frac{\xi^2\tau^2}{2}}\partial_\tau\left(\frac{e^{-\xi\mu}\Phi\left(-\xi\tau+\frac{\mu}{\tau}\right)-e^{\xi\mu}\Phi\left(-\xi\tau-\frac{\mu}{\tau}\right)}{1-2\Phi\left(-\frac{\mu}{\tau}\right)}\right) = 0$$

or

$$\frac{\frac{2\mu}{\tau^3}\phi\left(\frac{\mu}{\tau}\right)}{1-2\Phi\left(-\frac{\mu}{\tau}\right)} + \xi^2 = \frac{\left(\frac{\xi}{\tau}+\frac{\mu}{\tau^3}\right)e^{-\xi\mu}\phi\left(-\xi\tau+\frac{\mu}{\tau}\right)-\left(\frac{\xi}{\tau}-\frac{\mu}{\tau^3}\right)e^{\xi\mu}\phi\left(-\xi\tau-\frac{\mu}{\tau}\right)}{e^{-\xi\mu}\Phi\left(-\xi\tau+\frac{\mu}{\tau}\right)-e^{\xi\mu}\Phi\left(-\xi\tau-\frac{\mu}{\tau}\right)} \tag{A16}$$

Substitution of variables yields

$$\frac{1-2\Phi\left(-\frac{\mu}{\tau}\right)}{\frac{2\mu}{\tau^3}\phi\left(\frac{\mu}{\tau}\right)} = \int_\tau^\infty \frac{\tau^3}{u^2}e^{-\frac{\mu^2}{2u^2}+\frac{\mu^2}{2\tau^2}}du \equiv a(\tau)$$

for the inverse of the first term on the left-hand side of (A16), and, after reducing the numerator on the rights-hand side of (A16) to

$$\left(\frac{\xi}{\tau}+\frac{\mu}{\tau^3}\right)e^{-\xi\mu}\phi\left(-\xi\tau+\frac{\mu}{\tau}\right)-\left(\frac{\xi}{\tau}-\frac{\mu}{\tau^3}\right)e^{\xi\mu}\phi\left(-\xi\tau-\frac{\mu}{\tau}\right) \propto \frac{2\mu}{\sqrt{2\pi}\tau^3}e^{-\frac{\xi^2\tau^2}{2}-\frac{\mu^2}{2\tau^2}}$$

the inverse of the right-hand side becomes

$$\frac{e^{-\xi\mu}\Phi\left(-\xi\tau+\frac{\mu}{\tau}\right)-e^{\xi\mu}\Phi\left(-\xi\tau-\frac{\mu}{\tau}\right)}{-\frac{2\mu}{\sqrt{2\pi}\tau^3}e^{-\frac{\xi^2\tau^2}{2}-\frac{\mu^2}{2\tau^2}}} = \int_\tau^\infty \frac{\tau^3}{u^2}e^{-\frac{\xi^2u^2}{2}-\frac{\mu^2}{2u^2}+\frac{\xi^2\tau^2}{2}+\frac{\mu^2}{2\tau^2}}du \equiv b(\tau)$$

The first-order condition is then equivalent to

$$\frac{1}{a}+\xi^2=\frac{1}{b}\Leftrightarrow ab=\frac{a-b}{\xi^2} \tag{A17}$$

The second-order condition at any local maximum is thus

$$\frac{a'(\tau)}{a^2(\tau)}-\frac{b'(\tau)}{b^2(\tau)}<0$$

Substituting

$$a'(\tau)=\left(\frac{3}{\tau}-\frac{\mu^2}{\tau^3}\right)a(\tau)-\tau$$

and

$$b'(\tau)=\left(\frac{3}{\tau}-\frac{\mu^2}{\tau^3}+\xi^2\tau\right)b(\tau)-\tau$$

and rearranging yields

$$\begin{aligned}&\left(\frac{3}{\tau}-\frac{\mu^2}{\tau^3}\right)ab^2-\left(\frac{3}{\tau}-\frac{\mu^2}{\tau^3}+\xi^2\tau\right)a^2b+\tau a^2-\tau b^2\\&=\left(\tau+\left(\frac{\mu^2}{\tau^3}-\frac{3}{\tau}\right)a\right)(a-b)b<0\Leftrightarrow\left(\frac{3}{\tau}-\frac{\mu^2}{\tau^3}\right)a>\tau\Leftrightarrow a'>0\end{aligned} \tag{A18}$$

where the initial equality follows from (A17) and the penultimate equivalence follows from $a(\tau)>b(\tau)$ for all τ. At any interior minimum of a,

$$a''(\tau)=\left(-\frac{3}{\tau^2}+\frac{3\mu^2}{\tau^4}\right)a(\tau)-1>0\Leftrightarrow 2\mu^2>3\tau^2$$

and thus a can have at most one interior minimum followed by at most one interior maximum. In view of

$$\lim_{\tau\to0}a(\tau)=\lim_{\tau\to\infty}a(\tau)=\infty$$

the minimum exists but there cannot be any maximum, and so there exists some τ^* such that $a'>0$ if and only if $\tau>\tau^*$. Hence, $g(\tau)$ can also have at most one interior minimum followed by at most one interior maximum. To rule out a minimum and establish existence of the unique maximum, observe that

$$\begin{aligned}\lim_{\tau\to\infty}\hat{g}(\tau)&\propto\lim_{\tau\to\infty}\int_\tau^\infty\frac{1}{u^2}e^{-\frac{\xi^2u^2}{2}-\frac{\mu^2}{2u^2}}du\left(\int_\tau^\infty\frac{1}{u^2}e^{-\frac{\mu^2}{2u^2}-\frac{\xi^2\tau^2}{2}}du\right)^{-1}\\&=\lim_{\tau\to\infty}\left(1+\int_\tau^\infty\frac{\xi^2\tau^3}{u^2}e^{-\frac{\mu^2}{2u^2}+\frac{\mu^2}{2\tau^2}}du\right)^{-1}=0\end{aligned}$$

and

$$\lim_{\tau\to 0}\hat{g}(\tau) \propto e^{\xi\bar{X}} > 0$$

and that, after substituting back $\tau = \omega\sqrt{t - T_n}$,

$$\lim_{T_n\to t}\hat{g}'(T_n;t) \propto -r\hat{g}(T_n;t) < 0$$

Hence, $\hat{g}(T_n;t)$ is quasiconcave in T_n. To establish the local quasiconcavity claim, it remains to be shown that intervals of P exist on which $\hat{g}(P;t)$ is quasiconcave.

Consider now the earnings generated at time t from a given history of investments with initiation dates $\boldsymbol{T} = (T_1, \dots, T_n)$. As $\sigma \to 0$, the probability of abandonment vanishes for any investment, in view of

$$\lim_{\sigma\to 0}\bar{F}(t - T_i) = \lim_{\sigma\to 0} 2\Phi\left(\frac{\underline{Y} - \bar{X}}{\sigma\sqrt{t - T_i}}\right) = 0$$

and so

$$\lim_{\sigma\to 0}\Pr(n^* = n; t) = 1$$

for any n. The probability density of the cash flow of each of the n investments is given by

$$f(y;t) = \frac{\phi\left(\frac{y-\bar{X}}{\sigma\sqrt{t}}\right) - \phi\left(\frac{y-2\underline{Y}+\bar{X}}{\sigma\sqrt{t}}\right)}{\sigma\sqrt{t}}$$

The asymptotic likelihood ratio of any two cash flow vectors $\boldsymbol{y} = (y_1, \dots, y_n)$ and $\boldsymbol{z} = (z_1, \dots, z_n)$, with identical earnings P and identical investment dates T_i, becomes

$$\lim_{\sigma\to 0}\frac{\prod_{i=1}^{n} f(y_i;t)}{\prod_{i=1}^{n} f(z_i;t)} = \lim_{\sigma\to\infty} e^{\sum_{i=1}^{n}\frac{(z_i-\bar{X})^2-(y_i-\bar{X})^2}{2\sigma^2(t-T_i)}} \tag{A19}$$

where

$$y_n = P - y_1 - \cdots - y_{n-1}$$

and likewise for z_n. The quadratic form implies that there exists a unique $\boldsymbol{y}$ such that (A19) diverges for any $\boldsymbol{z} \neq \boldsymbol{y}$, with either $y_i \leq \bar{X}$ for all i or $y_i \geq \bar{X}$ for all i. The likelihood ratio of earnings P generated by n investments to earnings P generated by $n + 1$ investments, with identical investment dates T_1 through T_n and arbitrary T_{n+1}, is therefore divergent as $\sigma \to 0$ if and only if

$$\sum_{i=1}^{n}\frac{(y_i - \bar{X})^2}{t - T_i} < \sum_{i=1}^{n+1}\frac{(z_i - \bar{X})^2}{t - T_i} \tag{A20}$$

where $(y_1, \dots, y_n)$ and $(z_1, \dots, z_{n+1})$ are the respective maximum likelihood vectors. The earnings values that minimize either side of the inequality are $P = n\bar{X}$, in which case $y_i = \bar{X}$ for all i, and $P = (n+1)\bar{X}$, in which case $z_i = \bar{X}$ for all i. The quadratic form then implies that there exists a unique $P \in (n\bar{X}, (n-1)\bar{X})$ at which both sides are equal, and that the left-hand side of (A20) is increasing at this P while the right-hand side is increasing because $\bar{X} > 0$ for small σ, in view of Lemma 2. This observation applies to all possible sets of investment dates $\boldsymbol{T} = (T_1, \dots, T_{n+1})$. Thus, there exist cutoff values P_i, for $i = 1,2, \dots$, such that

$$\lim_{\sigma \to 0} \Pr\big(n = i \big| P \in (P_i, P_{i+1})\big) = 1$$

and so

$$\lim_{\sigma \to 0} \hat{g}(P \in (P_n, P_{n+1}); t) = \hat{g}(n\bar{X}; t) = \hat{g}(n; t)$$

The global claim then follows if it can be established that $\hat{g}(n; t)$ is quasiconcave in n. To this end, consider the joint probability

$$\begin{aligned} \Pr(T_n, n; t) &= \int_0^{T_n} \int_0^{T_n - T_1} \dots \int_0^{T_n - T_{n-2}} \prod_{i=1}^{n} \eta(T_i - T_{i-1}) H(t - T_n)\, dT_1 \dots dT_{n-1} \\ &= \frac{n\mu}{\omega\sqrt{T_n^3}} \phi\left(\frac{-n\mu}{\omega\sqrt{T_n}}\right) H(t - T_n) \end{aligned} \tag{A21}$$

where

$$\eta(t) = -H'(t) = \frac{\mu}{\omega\sqrt{t^3}} \phi\left(\frac{-\mu}{\omega\sqrt{t}}\right)$$

is the probability density of an investment at time t in the well-known form of an inverse Gaussian distribution, and

$$H(t) = 1 - \int_0^t \eta(s)\, ds = 1 - 2\Phi\left(\frac{-\mu}{\omega\sqrt{t}}\right)$$

The final equality in (A21) obtains because the n-fold convolution power of an inverse Gaussian distribution again yields an inverse Gaussian distribution (Barndorff-Nielsen 1978). After integration by parts, the expected option value becomes

$$\begin{aligned} \hat{g}(n; t) &= \int_0^t \hat{g}(T_n; t) \Pr(T_n; t|n)\, dT_n \\ &= \hat{g}(\bar{X} - \mu; t) - \int_0^t \hat{g}'(T_n; t) \Pr(T_n; t|n)\, dT_n \end{aligned} \tag{A22}$$

since $\Pr(X(t) = \bar{X} - \mu | T_n = t) = 1$, and where

$$\Pr(T_n; t|n) = \int_0^{T_n} \frac{n\mu}{\omega\sqrt{s^3}} \phi\left(\frac{-n\mu}{\omega\sqrt{s}}\right) H(t - s)\, ds \left(\int_0^t \frac{n\mu}{\omega\sqrt{s^3}} \phi\left(\frac{-n\mu}{\omega\sqrt{s}}\right) H(t - s)\, ds\right)^{-1}$$

The likelihood ratio

$$\frac{\partial_n \Pr(T_n, n; t)}{\Pr(T_n, n; t)} = 1 - \frac{3n^2\mu^2}{2\omega^2 T_n} \tag{A23}$$

is monotonically increasing in T_n. Hence, raising n shifts the relative weighting by (A23) monotonically to toward higher T_n. Further, (A23) is monotonically decreasing in n and has increasing differences in n and T_n, so that the weighting shift toward higher T_n accelerates monotonically in n. The quasi-concavity of the expected option value implies that there exists a unique $T_n^* \in (0, t)$ such that $-\hat{g}'(T_n; t) < 0$ if and only if $T_n < T_n^*$. Thus, if n is treated as continuous, there exists a unique n^* at which (A22) attains a stationary point, which must be a maximum since $\hat{g}(T_n; t) \geq 0$ and $\lim_{t\to\infty} \hat{g}(T_n; t) = 0$.

For the local claim, the above argument can be repeated with $\boldsymbol{y}$ and T_n. The likelihood ratio of cash flow vector $\boldsymbol{y} = (y_1, \dots, y_n)$ and investment history $\boldsymbol{T} = (T_1, \dots, T_n)$ is

$$\prod_{i=1}^{n} \frac{f'(y_i; t - T_i)}{f(y_i; t - T_i)} = \prod_{i=1}^{n} \frac{\bar{X} - y_i + (y_i - 2\underline{Y} + \bar{X})e^{-\frac{2(\bar{X}-\underline{Y})(y-\underline{Y})}{\sigma^2(t-T_i)}}}{2\sigma^2(t - T_i)\left(1 + e^{-\frac{2(\bar{X}-\underline{Y})(y-\underline{Y})}{\sigma^2(t-T_i)}}\right)} \tag{A24}$$

As $\sigma \to 0$, the exponential terms vanish fastest, which leaves the likelihood ratio increasing in T_i for all i if $y_i < \bar{X}$ and decreasing if $y_i > \bar{X}$, and with decreasing differences in y_i and T_i. As noted above, $y_i < \bar{X}$ obtains at the likelihood-maximizing cash flow vector if and only if $P < n\bar{X}$. Hence, $\hat{g}(P; t)$ is quasiconcave in the direction of T_n on $P \in (P_n, n\bar{X})$ and quasiconcave in the direction of $-T_n$ on $P \in (n\bar{X}, P_{n+1})$.

Proof of Proposition 2b The global claim follows from the observation in Corollary 1.1 that

$$\lim_{\omega\to 0} \Pr\big(n = i \big| P \in (i\underline{Y}, (i-1)\underline{Y})\big) = 1$$

and from the properties of $\hat{g}(n; t)$ established in Proposition 2a. The local claim follows from (A24) and from the properties of $\hat{g}(T_n; t)$ established in Proposition 2a.

Proof of Lemma 3. The proof will proceed with the conjecture that the optimal decision strategy takes the form of a two-sided threshold rule $\{\underline{X}, \bar{X}\}$ and subsequently verify that both thresholds are indeed unique. The identifying conditions for $\underline{X}$ and $\bar{X}$ are given by five equations:

$$\frac{\omega^2}{2} g''(X) - \kappa = rg(X) \tag{A25}$$

at all $X \in (\underline{X}, \bar{X})$, the value matching condition

$$g(\bar{X}) = g(\bar{X} - \mu) + v(\bar{X}) \tag{A26}$$

and the smooth pasting condition

$$g'(\bar{X}) = g'(\bar{X} - \mu) + v'(\bar{X}) \tag{A27}$$

for the investing threshold $\bar{X}$, and the value matching condition

$$g(\underline{X}) = 0 \tag{A28}$$

and the smooth-pasting condition

$$g'(\underline{X}) = 0 \tag{A29}$$

for the shut-down threshold $\underline{X}$. The general solution to (A25) is

$$g(X) = Ae^{\frac{\sqrt{2r}}{\omega}X} + Be^{-\frac{\sqrt{2r}}{\omega}X} - \frac{\kappa}{r} \tag{A30}$$

where A and B are unknown constants. After substitution of (A30) into the boundary conditions (A26) through (A29), solving (A26) and (A27) yields

$$A = \frac{v(\bar{X}) + \frac{\omega}{\sqrt{2r}} v'(\bar{X})}{2e^{\frac{\sqrt{2r}}{\omega}\bar{X}}\left(1 - e^{-\frac{\sqrt{2r}}{\omega}\mu}\right)} \tag{A31}$$

and

$$B = \frac{v(\bar{X}) - \frac{\omega}{\sqrt{2r}} v'(\bar{X})}{2e^{-\frac{\sqrt{2r}}{\omega}\bar{X}}\left(1 - e^{\frac{\sqrt{2r}}{\omega}\mu}\right)} \tag{A32}$$

and solving (A28) and (A29) yields

$$A = \frac{\kappa}{2r} e^{-\frac{\sqrt{2r}}{\omega}\underline{X}} \tag{A33}$$

and

$$B = \frac{\kappa}{2r} e^{\frac{\sqrt{2r}}{\omega}\underline{X}} \tag{A34}$$

After replacing the left-hand sides of (A31) and (A32) by (A33) and (A34), respectively, and then multiplying (A31) and (A32), one obtains

$$v(\bar{X}) = \sqrt{\frac{\omega^2}{2r}\left(v'(\bar{X})\right)^2 - \frac{\kappa^2}{r^2} e^{\frac{\sqrt{2r}}{\omega}\mu}\left(1 - e^{-\frac{\sqrt{2r}}{\omega}\mu}\right)^2} \tag{A35}$$

and

$$\underline{X} = \bar{X} - \mu - \frac{\omega}{\sqrt{2r}} \ln \frac{r}{\kappa} \frac{v(\bar{X}) + \frac{\omega}{\sqrt{2r}} v'(\bar{X})}{e^{\frac{\sqrt{2r}}{\omega}\mu} - 1} \tag{A36}$$

Equation (A35) has a unique solution if v is logarithmically concave. The going concern condition is met if the logarithmic term in (A36) is positive, which equates to

$$v(\bar{X}) \geq \frac{\kappa}{2r}\left(e^{\frac{\sqrt{2r}}{\omega}\mu} + e^{-\frac{\sqrt{2r}}{\omega}\mu} - 2\right) = g(\bar{X}|\bar{X} - \underline{X} = \mu)$$

after substitution of (A35) for g'.

Proof of Corollary 2.1. The distributional properties of n conditional on P and of T_n conditional on n continue to apply by the same arguments as in the proof of Proposition 2. Hence, the claim that $\hat{g}(P \in (P_n, P_{n+1}); t)$ is decreasing in n for large values of n obtains if it can be shown that $\hat{g}(T_n; t)$ is decreasing in T_n for T_n sufficiently close to t. To this end, determine first the probability density $f(x; t)$ of X at a future date t, given initial state $X = \bar{X} - \mu$. The density is characterized by the Fokker-Planck equation

$$f_t = \frac{\omega^2}{2} f_{xx}$$

with boundary condition

$$f(\underline{X}; t) = f(\bar{X}; t) = 0 \tag{A37}$$

for all t and initial condition

$$f(x; 0) = \delta(x - \bar{X} + \mu) \tag{A38}$$

where $\delta(\cdot)$ is the delta distribution. The solution is given by the Fourier series

$$f(x; t) = \sum_{n=1}^{\infty} \sin \frac{n\pi(\bar{X} - \mu - \underline{X})}{\bar{X} - \underline{X}} \sin \frac{n\pi(x - \underline{X})}{\bar{X} - \underline{X}} e^{-\left(\frac{n\pi}{\bar{X}-\underline{X}}\right)^2 \frac{\omega^2 t}{2}} \tag{A39}$$

The option value $\hat{g}(T_n; t)$ is computed conditional on the occurrence of neither investment nor shutdown through time t, but (A39) is an unconditional probability. To obtain the conditional probability, (A39) must be divided by the probability

$$\Pr(t < T^*) = \int_{\underline{X}}^{\bar{X}} f(x;t)\,dx$$
$$\propto \sum_{n=1}^{\infty} \sin\frac{n\pi(\bar{X}-\mu-\underline{X})}{\bar{X}-\underline{X}} e^{-\left(\frac{n\pi}{\bar{X}-\underline{X}}\right)^2\frac{\omega^2 t}{2}} \frac{\bar{X}-\underline{X}}{n\pi}(1-(-1)^n) \tag{A40}$$

that the firm has not been shut down or invested again by time t, where

$$T^* = \inf(s: X(s) = \underline{X} \vee X(s) = \bar{X})$$

The option value is thus

$$\hat{g}(T_n;t) = \int_{\underline{X}}^{\bar{X}} \frac{f(x;\tau)g(x)}{\Pr(t < \bar{T})}\,dx$$
$$\propto \sum_{n=1}^{\infty} \int_{\underline{X}}^{\bar{X}} \sin\frac{n\pi(\bar{X}-\mu-\underline{X})}{\bar{X}-\underline{X}} \sin\frac{n\pi(x-\underline{X})}{\bar{X}-\underline{X}} e^{-\left(\frac{n\pi}{\bar{X}-\underline{X}}\right)^2\frac{\omega^2\tau}{2}} \frac{e^{\frac{\sqrt{2r}}{\omega}(x-\underline{X})} + e^{\frac{\sqrt{2r}}{\omega}(\underline{X}-x)} - 2}{\Pr(t < T^*)}\,dx$$
$$= \sum_{n=1}^{\infty} \sin\frac{n\pi(\bar{X}-\mu-\underline{X})}{\bar{X}-\underline{X}} e^{-\left(\frac{n\pi}{\bar{X}-\underline{X}}\right)^2\frac{\omega^2\tau}{2}} \frac{\bar{X}-\underline{X}}{n\pi} \frac{2-(-1)^n\left(e^{\frac{\sqrt{2r}}{\omega}(\bar{X}-\underline{X})} + e^{\frac{\sqrt{2r}}{\omega}(\underline{X}-\bar{X})}\right)}{\left(1+\frac{2r}{\omega^2}\left(\frac{\bar{X}-\underline{X}}{n\pi}\right)^2\right)\Pr(t < T^*)} - 2$$

where $\tau \equiv t - T_n$. The claimed result obtains if $\hat{g}(T_n;t)$ is decreasing around $T_n = t$, or, equivalently, in increasing around $\tau = 0$. Indeed,

$$\partial_t \Pr(t < T^*) \propto \int_{\underline{X}}^{\bar{X}} f_{xx}(x;\tau)\,dx = \int_{\underline{X}}^{\bar{X}} \delta''(\bar{X}-\mu)\,dx = 0$$

when $\tau = 0$, and, similarly,

$$\int_{\underline{X}}^{\bar{X}} f_t(x;\tau)g(x)\,dx \propto \int_{\underline{X}}^{\bar{X}} f_{xx}(x;\tau)g(x)\,dx = \int_{\underline{X}}^{\bar{X}} \delta''\big(g(\bar{X}-\mu)\big)\,dx$$
$$= g''(\bar{X}-\mu) > 0 \tag{A41}$$

in view of the initial condition (A38). The final inequality in (A41) follows from the convexity of g. Hence, $\hat{g}(T_n;t)$ becomes decreasing in T_n as $T_n \to t$.

For the quasiconcavity claims, note that the case $\kappa = 0$ is identical to the result in Proposition 2a. In the converse case $\kappa \to \bar{\kappa}$, $\bar{X} - \mu \to \underline{X}$ and so the expected option value, conditional on the final investment date T_n, becomes

$$\lim_{\kappa\to\bar{\kappa}} \hat{g}(T_n;t) \propto \lim_{\varepsilon\to 0} \sum_{n=1}^{\infty} \sin\frac{n\pi\varepsilon}{\mu} e^{-\left(\frac{n\pi}{\mu}\right)^2\frac{\omega^2\tau}{2}} \frac{\mu}{n\pi} \frac{2-(-1)^n\left(e^{\frac{\sqrt{2r}}{\omega}\mu} + e^{-\frac{\sqrt{2r}}{\omega}\mu}\right)}{\left(1+\frac{2r}{\omega^2}\left(\frac{\mu}{n\pi}\right)^2\right)\Pr(t < T^*)} - 2 \tag{A42}$$

where

$$\Pr(t < T^*) \propto \sum_{n=1}^{\infty} \sin\frac{n\pi\varepsilon}{\mu} e^{-\left(\frac{n\pi}{\mu}\right)^2 \frac{\omega^2\tau}{2}} \frac{\mu}{n\pi}(1 - (-1)^n)$$

and $\tau = t - T_n$. Observe that $\hat{g}(T_n; t)$ is a weighted average of the fraction

$$q(n) = \frac{2 - (-1)^n \left(e^{\frac{\sqrt{2r}}{\omega}(\bar{X} - \underline{X})} + e^{\frac{\sqrt{2r}}{\omega}(\underline{X} - \bar{X})}\right)}{\left(1 + \frac{2r}{\omega^2}\left(\frac{\bar{X} - \underline{X}}{n\pi}\right)^2\right)\Pr(t < T^*)}$$

with weighting terms

$$z(n) \equiv \sin\frac{n\pi\varepsilon}{\mu} e^{-\left(\frac{n\pi}{\mu}\right)^2 \frac{\omega^2\tau}{2}} \frac{\mu}{n\pi}$$

The fraction term q alternates in sign as n increases, with positive values for odd n and negative values for even n, and its magnitude increases monotonically in n. For ε sufficiently small, the $\sin(\cdot)$ term in z becomes positive for $n = 1, \ldots, N$ and arbitrarily large N, and hence the $z(n)$ are asymptotically non-negative for $n = 1, .., N$ as $\varepsilon \to 0$. As τ increases, the weights $\frac{z(n)}{\sum_{i=1}^{\infty} z(i)}$ shift monotonically toward lower n and thus toward positive values in the series. Therefore, $\hat{g}(T_n; t)$ becomes increasing in τ and, equivalently, decreasing in T_n as $\kappa \to \bar{\kappa}$. The claim with respect to P now follows by the same argument as in Proposition 2a.

References

Abel, A., Dixit, A., Eberly, J., & Pindyck, R. (1996). Options, the Value of Capital, and Investment. *The Quarterly Journal of Economics, 111*(3), 753-777.

Ashton, D., Cooke, T., & Tippett, M. (2003). An Aggregation Theorem for the Valuation of Equity Under Linear Information Dynamics. *Journal of Business Finance & Accounting, 30*(3), 413-440.

Ataullah, A., Rhys, H., & Tippett, M. (2009). Non-linear Equity Valuation. *Accounting and Business Research, 39*(1), 57-73.

Baldenius, T., Nezlobin, A., & Vaysman, I. (2016). Managerial Performance Evaluation and Real Options. *The Accounting Review, 91*(3), 741-766.

Barndorff-Nielsen, O. (1978). Hyperbolic Distributions and Distributions on Hyperbolae. *Scandinavian Journal of Statistics, 5*(3), 151-157.

Barth, M., Beaver, W., & Landsman, W. (1998). Relative Valuation Roles of Equity Book Value and Net Income as a Function of Financial Health. *Journal of Accounting and Economics, 25*(1), 1-34.

Berger, P., Ofek, E., & Swary, I. (1996). Investor Valuation of the Abandonment Option. *Journal of Financial Economics, 42*(2), 257-287.

Berk, J., Green, R., & Naik, V. (1999). Optimal Investment, Growth Options, and Security Returns. *Journal of Finance, 54*(5), 1553-1607.

Brooks, L., & Buckmaster, D. (1976). Further Evidence of the Time Series Properties of Accounting Income. *Journal of Finance, 31*(5), 1359-1373.

Burgstahler, D., & Dichev, I. (1997). Earnings, Adaptation and Equity Value. *The Accounting Review, 72*(2), 187-215.

Chen, P., & Zhang, G. (2007). How Do Accounting Variables Explain Stock Price Movements? Theory and Evidence. *Journal of Accounting and Economics, 43*(2), 219-244.

Collins, D., Pincus, M., & Xie, H. (1999). Equity Valuation and Negative Earnings: The Role of Book Value of Equity. *The Accounting Review, 74*(1), 29-61.

Dechow, P., Ge, W., & Schrand, C. (2010). Understanding Earnings Quality: A Review of the Proxies, their Determinants and their Consequences. *Journal of Accounting and Economics, 50*(2), 344-401.

Dixit, A. (1989). Entry and Exit Decisions under Uncertainty. *Journal of Political Economy, 97*(3), 620-638.

Dixit, A., & Pindyck, R. (1994). *Investment under Uncerainty.* Princeton: Princeton University Press.

Feltham, G., & Ohlson, J. (1995). Valuation and Clean Surplus Accounting for Operating and Financial Activities. *Contemporary Accounting Research, 11*(2), 689-731.

Feltham, G., & Ohlson, J. (1996). Uncertainty Resolution and the Theory of Depreciation Measurement. *Journal of Accounting Research, 34*(2), 209-234.

Freeman, R., & Tse, S. (1992). A Nonlinear Mode of Security Price Responses to Unexpected Earnings. *Journal of Accounting Research, 30*(2), 185-209.

Gietzmann, M., & Ostaszewski, A. (2004). Predicting Firm Value: the Superiority of q-Theory over Residual Income. *Accounting and Business Research, 34*(4), 349-377.

Govindaraj, S. (1992). Linear Valuation Models in Continuous Time with Accounting Information. *Journal of Accounting, Auditing & Finance, 7*(4), 485-507.

Hayn, C. (1995). The Information Content of Losses. *Journal of Accounting and Economics, 20*(1), 125-153.
Hemmer, T., & Labro, E. (2016). *Management by the Numbers: A Formal Approach to Deriving Informational and Distributional Properties of 'Un-managed' Earnings*. Working paper, University of North Carolina.
Joos, P., & Plesko, G. (2005). Valuing Loss Firms. *The Accounting Review, 80*(3), 847-870.
Kothari, S. (2001). Capital Markets Research in Accounting. *Journal of Accounting and Economics, 31*(1), 105-231.
Livdan, D., & Nezlobin, A. (2016). *Accounting Rules, Equity Valuation, and Growth Options*. Working Paper, University of California, Berkely.
McDonald, R., & Siegel, D. (1985). Investment and the Valuation of Firms When There is an Option to Shut Down. *International Economic Review, 26*(2), 331-349.
McDonald, R., & Siegel, D. (1986). The Value of Waiting to Invest. *The Quarterly Journal of Economics, 101*(4), 707-728.
Ohlson, J. (1995). Earnings, Book Values, and Dividends in Equity Valuation. *Contemporary Accounting Research, 11*(2), 661-687.
Penno, M. (1996). Unobservable Precision Choices in Financial Reporting. *Journal of Accounting Research, 34*(1), 141-149.
Ross, S. (1996). *Stochastic Processes*. New York: John Wiley & Sons.
Shanthikumar, G., & Yao, D. (1986). The Preservation of Likelihood Ratio Ordering Under Convolution. *Stochastic Processes and their Applications, 23*(2), 259-267.
Subramanyam, K. (1996). Uncertain Precision and Price Reactions to Information. *The Accounting Review, 71*(2), 207-219.
Zhang, G. (2000). Accounting Information, Capital Investment Decisions, and Equity Valuation: Theory and Empirical Implications. *Journal of Accounting Research, 38*(2), 271-295.

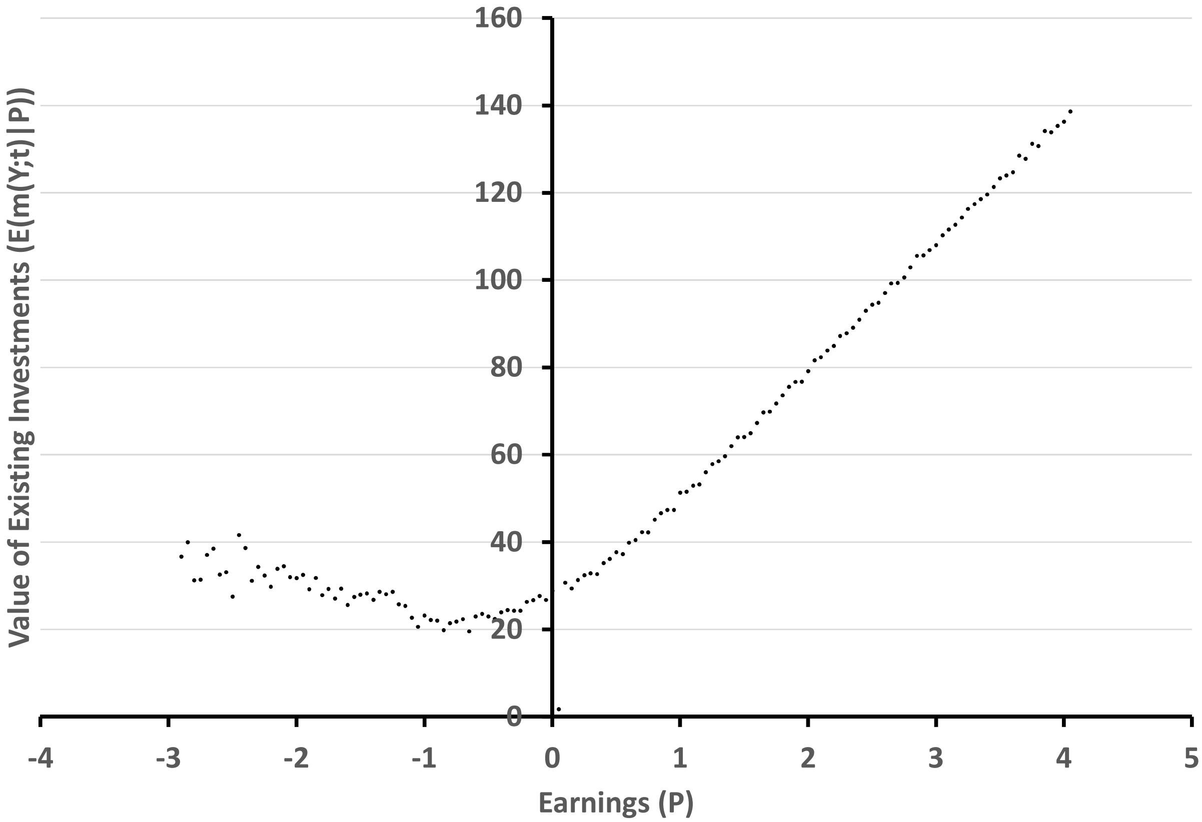

Figure 1. The value of existing investments, $\widehat{m}(P;t) = E(m(\boldsymbol{Y};t)|P)$, as a function of earnings, based on a simulation of 700,000 firms, with parameter settings $\sigma = 0.4$, $\omega = 0.3$, $\mu = 0.1$, $r = 0.05$ and $t = 500$. Continuous-time stochastic processes are approximated by discrete increments of size 0.01. Plotted values are averages over intervals of length 0.05.

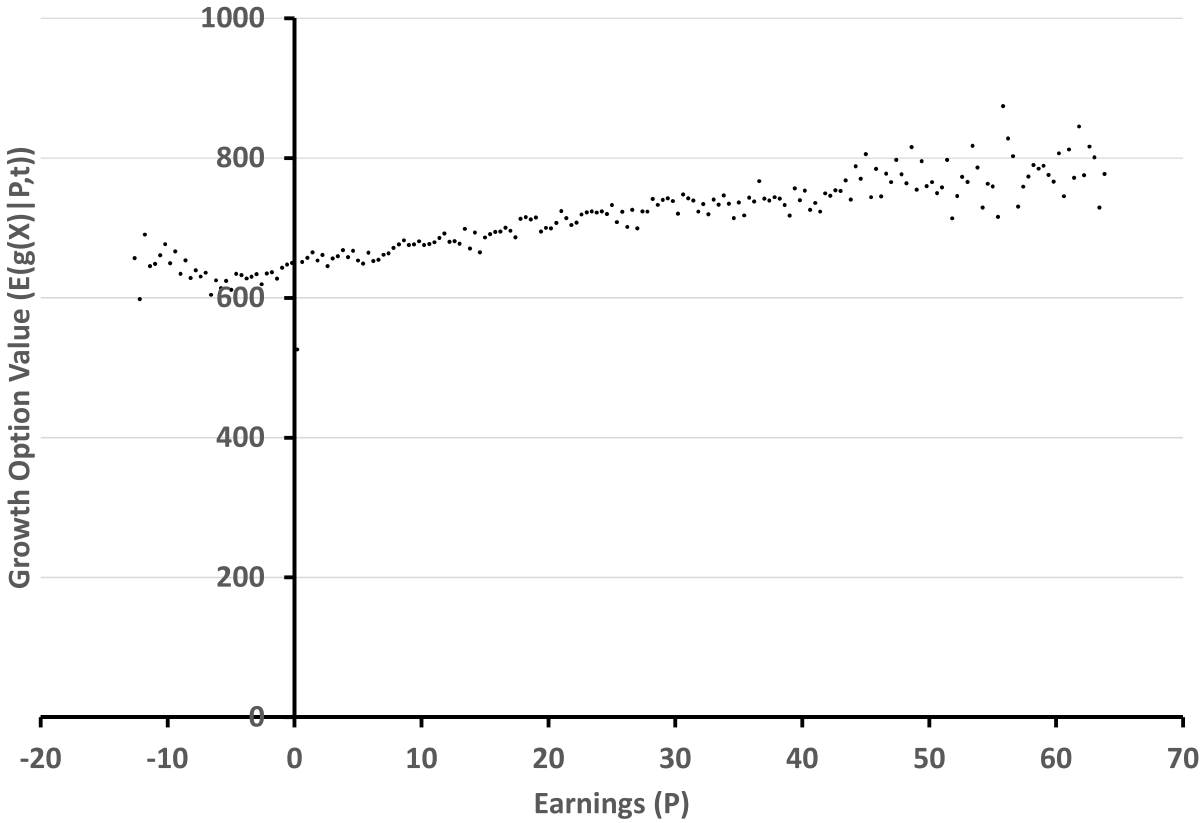

Figure 2. The value of future growth opportunities, $\hat{g}(P;t) = E(g(X)|P,t)$, as a function of earnings, based on a simulation of 100,000 firms, with parameter settings $\sigma = 3$, $\omega = 2$, $\mu = 0.5$, $r = 0.05$ and $t = 500$. Continuous-time stochastic processes are approximated by discrete increments of size 0.4.

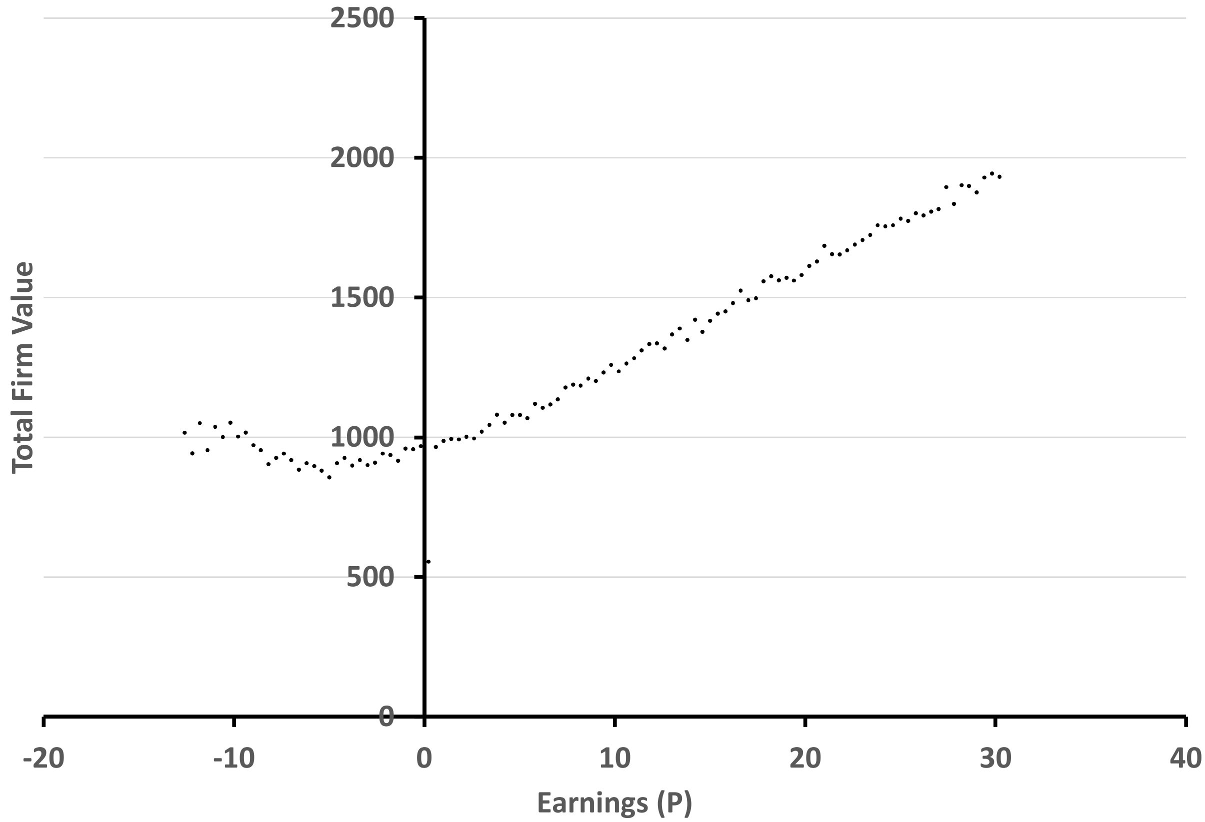

Figure 3. Total firm value, $\hat{m}(P;t) + \hat{g}(P;t)$, as a function of earnings, based on a simulation of 100,000 firms, with parameter settings $\sigma = 3$, $\omega = 2$, $\mu = 0.5$, $r = 0.05$ and $t = 500$. Continuous-time stochastic processes are approximated by discrete increments of size 0.4.